Electromagnetic Interactions[1]

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Contents

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[1][Feynman 21-6]

Chapter 1

Maxwell's Equations and Electromagnetic Fields

1.1 Introduction

1.1.1 Maxwell's Equations (1865)

The governing equations of electromagnetism

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla.\mathbf{B} = 0$$

(Coulomb/Gauss's Law)

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

(Faraday's Law)   (Ampere's Law)

$$\tag{1.1}$$

$\mathbf{E}$ electric field, describes the force felt by a (stationary) charge q: $\mathbf{F} = q\mathbf{E}$

$\mathbf{B}$ magnetic field, describes the force felt by a current

i.e. a moving charge (velocity $\mathbf{v}$): $\mathbf{F} = q\mathbf{v} \wedge \mathbf{B}$

Thus the *Lorentz Force* (on charge q) is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) \tag{1.2}$$

ρ electric charge density (Coulombs/m^3). Total charge $Q = \int_V \rho d^3 x$

$\mathbf{j}$ electric current density (Coulombs/s/m^2)

Current crossing area element $\mathbf{dA}$ is $\mathbf{j} . \mathbf{dA}$ Coulomb/s = Amps.

[Note. Sometimes the first Maxwell equation is called Gauss's rather than Coulomb's Law.]
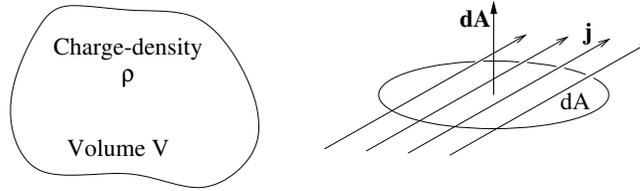
Figure 1.1: Charge density is local charge per unit volume. Current density is current per unit area.

1.1.2 Historical Note

Much scientific controversy in 2nd half of 19th century concerned question of whether $\mathbf{E}$, $\mathbf{B}$ were 'real' physical quantities of science or else mere mathematical conveniences for expressing the forces that charges exert on one another. English science (Faraday, Maxwell) emphasized the fields; German mostly the act-at-a-distance. Since ~ 1900 this question has been regarded as settled in favor of the fields. And modern physics, if anything, tends to regard the field as more fundamental than the particle.

1.1.3 Auxiliary Fields and Electromagnetic Media

Electromagnetic texts often discuss two additional "auxiliary" fields $\mathbf{D}$ the "electric displacement" and $\mathbf{H}$ the "magnetic intensity" which account for dielectric and magnetic properties of materials. These fields are not fundamental and introduce unnecessary complication and possible confusion for most of our topics. Therefore we will avoid them as much as possible. For the vacuum, $\epsilon_0 \mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mu_0 \mathbf{H}$.

1.1.4 Units

Historically there were two (or more!) <u>different</u> systems of units, one defining the quantity of charge in terms of the force between two stationary <u>charges</u> (the "Electrostatic" units) and one defining it in terms of forces between (chargeless) <u>currents</u> (the "Electromagnetic" system). Electrostatic units are based on Coulomb's law $\nabla.\mathbf{E} = \rho/\epsilon_0$ and electromagnetic units on the (steady-state version of) Ampere's law $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$. The quantities $1/\epsilon_0$ and μ_0 are therefore fundamentally calibration factors that determine the size of the unit charge. Choosing one or other of them to be 4π amounts to choosing electrostatic or electromagnetic units. However, with the unification of electromagnetism, and the subsequent realization that the speed of light is a fundamental constant, it became clear that the units of electromagnetism ought to be defined in terms of only one of these laws and the speed of light. Therefore the "System Internationale" SI (or sometimes MKSA) units adopts the electromagnetic definition because it can be measured most easily, but with a different μ_0, as follows. "One Ampere is that current which, when flowing in two infinitesimal parallel wires 1m apart produces a force of 2×10^{-7} Newtons per meter of their length." An Amp is one Coulomb per second. So this defines the unit of charge. We will show later that this

definition amounts to defining

$$\mu_0 = 4\pi \times 10^{-7} \qquad \text{(Henry/meter)} \tag{1.3}$$

and that because the ratio of electromagnetic to electrostatic units is c^2

$$\epsilon_0 = \frac{1}{c^2 \mu_0} = 8.85 \times 10^{-12} \qquad \text{(Farad/meter)} \tag{1.4}$$

μ_0 is called the "permeability of free space". ϵ_0 is called the "permittivity of free space". [See J.D. Jackson 3rd Ed, Appendix for a detailed discussion.]

1.2 Vector Calculus and Notation

Electromagnetic quantities include vector fields $\mathbf{E}, \mathbf{B}$ etc. and so EM draws heavily on vector calculus. ∇ is shorthand for a vector operator (gradient)

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \frac{\partial \phi}{\partial x_i} \quad \text{(suffix notation)} \tag{1.5}$$

giving a vector gradient from a scalar field ϕ. ∇ can also operate on vector fields by scalar (.) or vector ($\wedge$) multiplication.

1.2.1 Divergence

$$\nabla . \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_i}{\partial x_i} \tag{1.6}$$

1.2.2 Curl

$$\nabla \wedge \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = \epsilon_{ijk} \frac{\partial E_k}{\partial x_j} \tag{1.7}$$

1.2.3 Volume Integration

$$\int_V d^3 x \tag{1.8}$$

$d^3 x$ is shorthand for $dxdydz = dV$, the volume element.

1.2.4 Surface Integration

$$\int_S \mathbf{v} . \mathbf{dA} \quad . \tag{1.9}$$

The surface element $\mathbf{dA}$ or often $\mathbf{dS}$ is a vector normal to the element.

dA

dA

dl

Surface Element   Line Element
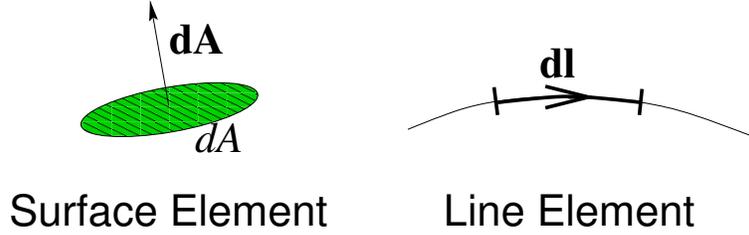
Figure 1.2: Elements for surface and line integrals.

1.2.5 Line (Contour) Integration

$$\int_C \mathbf{v}.\mathbf{dl} \quad . \tag{1.10}$$

Line element $\mathbf{dl}$.

1.2.6 The Meaning of divergence: ∇.

dy

dz

dx
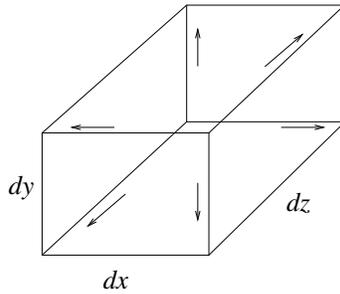
Figure 1.3: Cartesian volume element.

Consider a volume element. Evaluate the total flux of a vector field $\mathbf{v}$ out across the element's surface. It is the sum $\mathbf{v}.\mathbf{dA}$ over the six faces of the cuboid

$$
\begin{aligned}
\left[v_x\left(x+dx/2\right)-v_x\left(x-dx/2\right)\right] dydz+ & \\
\left[v_y\left(y+dy/2\right)-v_y\left(y-dy/2\right)\right] dzdx+ & \\
\left[v_z\left(z+dz/2\right)-v_z\left(z-dz/2\right)\right] dxdy \;=\; & \\
dxdydz\left[\frac{dv_x}{dx}+\frac{dv_y}{dy}+\frac{dv_z}{dz}\right] \;=\;\; & d^3x\nabla.\mathbf{v}=dV\nabla.\mathbf{v}
\end{aligned}
\tag{1.11}
$$

So for this elemental volume:

$$\int_{dS}\mathbf{v}.\mathbf{dA}=\int_{dV}\nabla.\mathbf{v}\,d^3x \tag{1.12}$$

But any arbitrary finite volume can be considered to be the sum of many small cuboidal elements. Adjacent internal face contribution cancel out hence only the external surface contributions remain, so

$$\int_S\mathbf{v}.\mathbf{dA}=\int_V\nabla.\mathbf{v}\,d^3x \tag{1.13}$$

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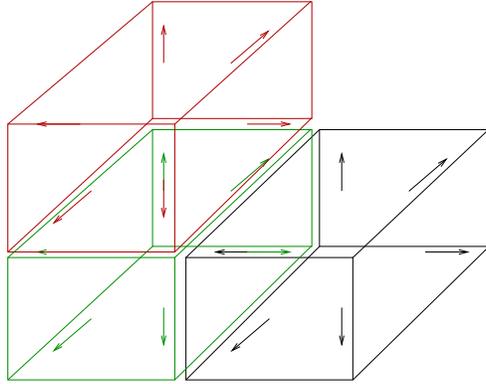
Figure 1.4: Adjacent faces cancel out in the sum of divergence from many elements.

for <u>any</u> volume V with surface S, and arbitrary vector field $\mathbf{v}$. This is <u>Gauss's Theorem</u>.

1.2.7 The Meaning of Curl: $\nabla\wedge$
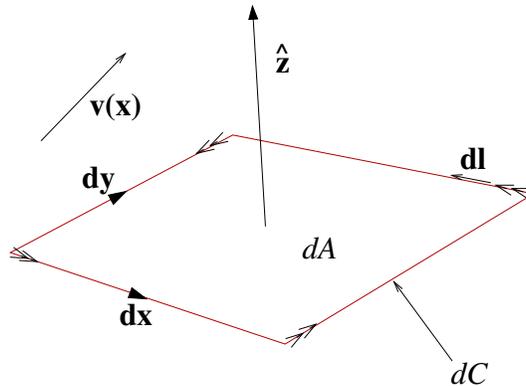
Figure 1.5: Rectangular surface element with axes chosen such that the normal is in the $\mathbf{z}$-direction.

Consider an arbitrary rectangular <u>surface</u> element Choose axes such that normal is in z-direction and edges along $\mathbf{x}$ and $\mathbf{y}$. Arbitrary vector field $\mathbf{v}(\mathbf{x})$. Evaluate the contour integral of $\mathbf{v}$, clockwise around dC (relative to the direction $\hat{\mathbf{z}}$), round boundary of element, regarding the center as $\mathbf{x}$.

$$
\begin{aligned}
\oint_{dC} \mathbf{v}.\mathbf{dl} &= \mathbf{v}(x, y - dy/2).\mathbf{dx} + \mathbf{v}(x + dx/2, y).\mathbf{dy} \\
&+ \mathbf{v}(x, y + dy/2).(-\mathbf{dx}) + \mathbf{v}(x - dx/2, y).(-\mathbf{dy}) \\
&= -dv_x dx + dv_y dy = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) dx dy \\
&= (\nabla \wedge \mathbf{v})_z dA = (\nabla \wedge \mathbf{v}).\mathbf{dA} \quad .
\end{aligned}
$$

(1.14)

(1.15)

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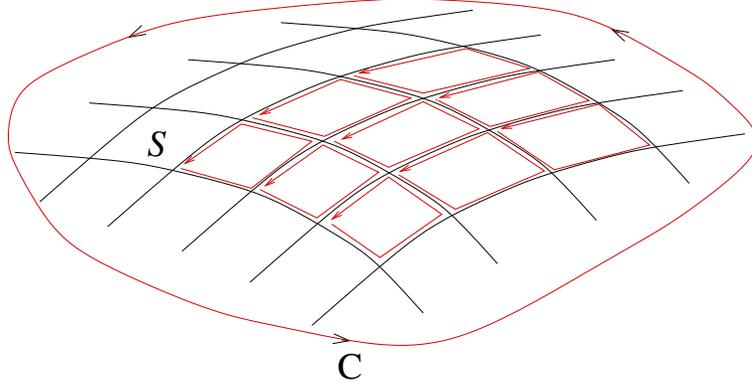
Figure 1.6: Arbitrary surface may be divided into the sum of many rectangular elements. Adjacent edge integral contributions cancel.

So integral $\mathbf{v.dl}$ around element is equal to the curl scalar-product area element. Apply to arbitrary surface; divide surface up into many elements dA. All <u>internal</u> edge integrals cancel. Hence

$$\oint_C \mathbf{v.dl} = \int_S (\nabla \wedge \mathbf{v}) \, \mathbf{.dA} \tag{1.16}$$

This is <u>Stokes' Theorem</u>.

1.3 Electrostatics and Gauss' Theorem

Gauss's theorem is the key to understanding electrostatics in terms of Coulomb's Law $\nabla.\mathbf{E} = \rho/\epsilon_0$.

1.3.1 Point Charge q

Apply Gauss's Theorem to a sphere surrounding q

$$\int_S \mathbf{E.dA} = \int_V \nabla.\mathbf{E} \, d^3x = \int_V \frac{\rho}{\epsilon_0} \, d^3x = \frac{q}{\epsilon_0} \quad . \tag{1.17}$$

But by spherical symmetry $\mathbf{E}$ must be in radial direction and E_r has magnitude constant over the sphere. Hence $\int_S \mathbf{E.dA} = \int_S E_r dA = E_r \int_S dA = E_r 4\pi r^2$. Thus

$$E_r 4\pi r^2 = \frac{q}{\epsilon_0} \qquad \text{or} \tag{1.18}$$

$$E_r = \frac{q}{4\pi\epsilon_0 r^2} \qquad \text{i.e.} \qquad \mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} \quad . \tag{1.19}$$

Consequently, force on a second charge at distance r is

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \tag{1.20}$$

Coulomb's inverse-square-law of electrostatic force.
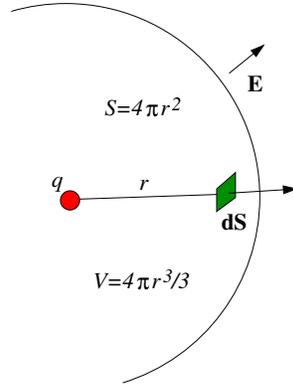
Figure 1.7: Spherical volume, V over which we perform an integral of Coulomb's law to deduce **E**.

1.3.2 Spherically Symmetric Charge $\rho(r)$

Notice that point-charge derivation depended only on symmetry. So for a distributed charge-density that is symmetric argument works just the same i.e.

$$\mathbf{E} = \hat{\mathbf{r}} E_r \qquad E_r = \frac{q}{4\pi\epsilon_0 r^2} \tag{1.21}$$

$$\text{where now} \quad q = \int_V \rho \, d^3x = \int_0^r \rho(r) 4\pi r^2 dr. \tag{1.22}$$

Electric field due to a spherically symmetric charge density is equal to that of a point charge of magnitude equal to the total charge within the radius, placed at the spherical center.

1.3.3 Arbitrary Charge Distribution

If there is no specific symmetry Gauss's Theorem still applies:
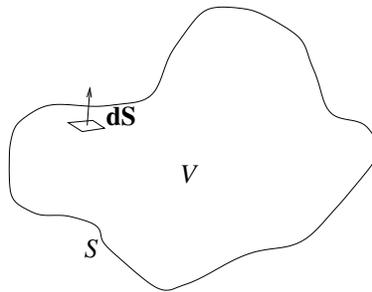
Figure 1.8: Arbitrary volume for Gauss's Theorem.

$$\int_S \mathbf{E}.\mathbf{dA} = \int_V \nabla.\mathbf{E} \, d^3x = \int_V \frac{\rho}{\epsilon_0} \, d^3x = \frac{q}{\epsilon_0} \tag{1.23}$$

q is the total charge (integral of charge density) over the volume. $\int_S \mathbf{E}.\mathbf{dA}$ is the total flux of electric field across the surface S.

1.3.4 Intuitive Picture
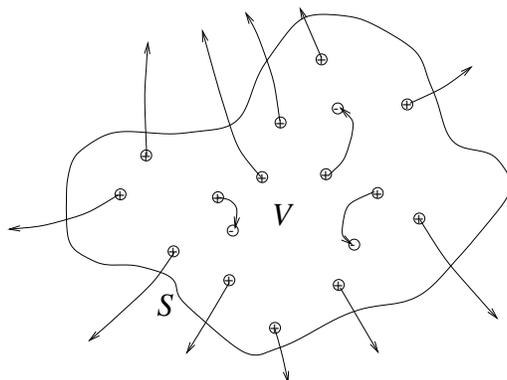
Figure 1.9: Intuitive picture of charges and field-lines.

Each ($+ve$) charge is the origination point of an electric-field-line. [Each $-ve$ charge is termination of a field line]. The total charge in volume V determines the number of field-lines that start in V. Field lines only start/end on charges (Coulomb's Law) so all must escape from the volume, crossing surface S (somewhere). [Field lines that start <u>and</u> end in V contribute neither to $\int \mathbf{E.dA}$ nor to q, because of cancellation]. $\int_S \mathbf{E.dA}$ can be thought of as counting the "number of field lines" crossing the surface. [Of course it is an arbitrary choice how big we consider the charge is that gives rise to one field-line.] Intuitive view of electric field "intensity": Strength of E is proportional to the number of field-lines per unit area. (Fig 1.10). All these intuitive views are conceptually helpful but are not formally necessary.

Field-Lines and Field Intensity.
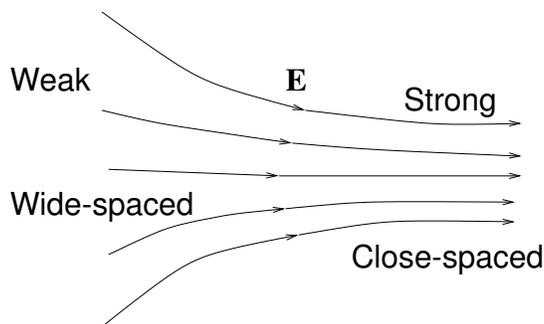
Figure 1.10: Spacing of field-lines is inversely proportional to field-strength.

Electromagnetism is considered completely described by Maxwell's equations <u>without</u> need for these pictures.

1.3.5 Electric Potential (for static problems $\frac{\partial}{\partial t} \to 0$)

In the static situation there is no induction and Faraday's law becomes $\nabla \wedge \mathbf{E} = 0$. By the way, this equation could also be derived from the inverse-square-law by noting that
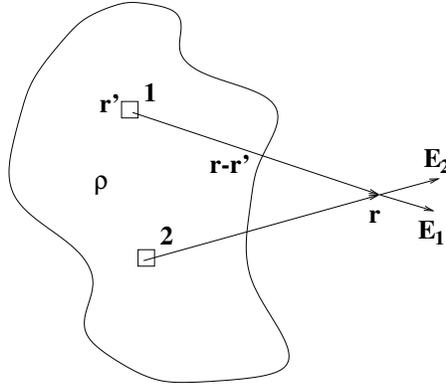
Figure 1.11: Each element contributes an irrotational component to $\mathbf{E}$. Therefore the total $\mathbf{E}$ is irrotational.

$$\nabla \wedge \frac{\mathbf{r}}{r^3} = 0 \tag{1.24}$$

so by the linearity of the $\nabla \wedge$ operator the sum (integral) of all Electric field contributions from any charge distribution is curl-free "irrotational":

$$\nabla \wedge \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' = 0 \tag{1.25}$$

[This shows that the spherical symmetry argument only works in the absence of induction $\dot{\mathbf{B}}$ would define a preferred direction; asymmetric!] For any vector field $\mathbf{E}$, $\nabla \wedge \mathbf{E} = 0$ is a necessary and sufficient condition that $\mathbf{E}$ can be written as the gradient of a scalar $\mathbf{E} = -\nabla\phi$.

Necessary

$$(\nabla \wedge \nabla\phi)_z = \frac{\partial}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x} = 0 \qquad \text{(et sim x, y)} \quad . \tag{1.26}$$

Curl of a gradient is zero.

Sufficient (prove by construction)

Apply Stokes' theorem to a closed contour consisting of any 2 paths between points 0 and $\mathbf{x}$.

$$\oint_C \mathbf{E}.\mathbf{dl} = \underbrace{\int_0^{\mathbf{x}} \mathbf{E}.\mathbf{dl}}_{\text{Path 2}} - \underbrace{\int_0^{\mathbf{x}} \mathbf{E}.\mathbf{dl}}_{\text{Path 1}} = \int_S \nabla \wedge \mathbf{E}.\mathbf{dS} \underset{\text{by hypothesis.}}{=0} \tag{1.27}$$
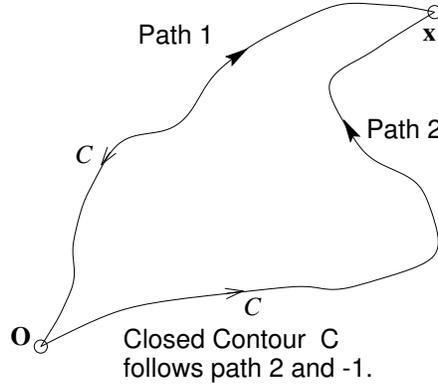
Figure 1.12: Two different paths from 0 to **x** construct a closed contour when one is reversed.

So $\nabla \wedge \mathbf{E} = 0 \Rightarrow \int_o^{\mathbf{x}} \mathbf{E}.\mathbf{dl}$ is independent of chosen path, i.e. it defines a unique[1] quantity. Call it $-\phi(\mathbf{x})$. Consider $\nabla\phi$ defined as the limit of $\delta\phi$ between adjacent points.

$$- \nabla\phi = \nabla \left(\int^{\mathbf{x}} \mathbf{E}.\mathbf{dl} \right) = \mathbf{E} \qquad . \tag{1.28}$$

Many electrostatic problems are most easily solved in terms of the electric potential ϕ because it is a scalar (so easier). Governing equation:

$$\nabla.\mathbf{E} = -\nabla.\nabla\phi = -\nabla^2\phi = \frac{\rho}{\epsilon_0} \tag{1.29}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \qquad \text{is the "Laplacian" operator} \tag{1.30}$$

$$\nabla^2\phi = \frac{-\rho}{\epsilon_0} \qquad \text{"Poisson's Equation".} \tag{1.31}$$

1.3.6   Potential of a Point Charge [General Potential Solution]

One can show by direct differentiation that

$$\nabla\frac{1}{r} = -\frac{\mathbf{r}}{r^3} \qquad . \tag{1.32}$$

So by our previous expression $\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$ we can identify

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \tag{1.33}$$

as the potential of a charge q (at the origin $\mathbf{x} = 0$).

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[1]The potential can always be altered by adding a constant, without changing **E**. This freedom can be considered equivalent to choosing the origin at which $\phi = 0$.

1.3.7 Green Function for the Laplacian

For a linear differential operator, $\mathcal{L}$, mathematicians define something called "Green's function" symbolically by the equation

$$\mathcal{L}G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \qquad . \tag{1.34}$$

If we can solve this equation in general, then solutions to

$$\mathcal{L}\phi = \rho(\mathbf{x}) \tag{1.35}$$

can be constructed for <u>arbitrary</u> ρ as

$$\phi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3x' \tag{1.36}$$

because of the (defining) property of the δ-function

$$\int f(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')\, d^3x' = f(\mathbf{x}) \qquad . \tag{1.37}$$

When $\mathcal{L}$ is the Laplacian, ∇^2, the Green function is

$$\boxed{G(\mathbf{x}, \mathbf{x}') = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|} \qquad .} \tag{1.38}$$

This fact may be derived directly from the solution for the potential for a point charge. Indeed, a point charge is exactly the delta-function situation whose solution is the Green function. In other words, the charge density for a point charge of magnitude q at position $\mathbf{x}'$ is

$$\rho(\mathbf{x}) = q\delta(\mathbf{x} - \mathbf{x}') \qquad ; \tag{1.39}$$

so the point-charge potential, namely,

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x} - \mathbf{x}'|} \qquad , \tag{1.40}$$

is the solution of the equation:

$$\nabla^2\phi = -\frac{q}{\epsilon_0}\delta(\mathbf{x} - \mathbf{x}') \qquad . \tag{1.41}$$

Consequently, solution of Poisson's equation can be written as the integral of the Green function:

$$\boxed{\phi(\mathbf{x}) = \int \left(\frac{-\rho(\mathbf{x}')}{\epsilon_0}\right)\left(\frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|}\right) d^3x' = \int \frac{\rho(\mathbf{x}')}{4\pi\epsilon_0}\frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \qquad .} \tag{1.42}$$

Informally, the smooth charge distribution ρ can be approximated as the sum ($\to \int$) of many point charges $\rho(x')\, d^3x'$, and the potential is the sum of their contributions.

1.3.8 Boundary Conditions

Strictly speaking, our solution of Poisson's equation is not unique. We can always add to ϕ a solution of the homogeneous (Laplace) equation $\nabla^2\phi = 0$. The solution only becomes unique when boundary conditions are specified. The solution

$$\phi(\mathbf{x}) = \int \frac{\rho\left(x'\right)}{4\pi\epsilon_0} \frac{d^3 x'}{|\mathbf{x} - \mathbf{x}'|} \tag{1.43}$$

is correct when the boundary conditions are that

$$\phi \to 0 \qquad \text{as} \qquad |\mathbf{x}| \to \infty \quad : \tag{1.44}$$

no applied external field.

In practice most interesting electrostatic calculations involve specific boundaries. A big fraction of the work is solving Laplace's equation with appropriate boundary conditions. These are frequently the specification of ϕ on (conducting) surfaces. The charge density on the conductors is rarely specified initially.

1.3.9 Parallel Plate Capacitor
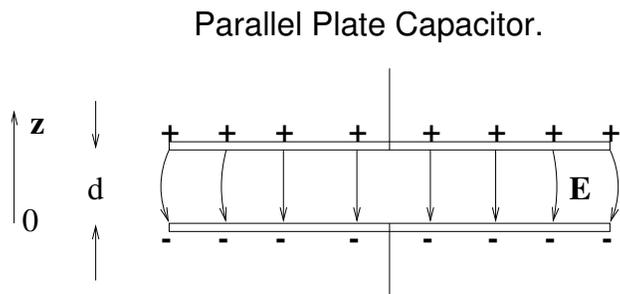
Figure 1.13: The parallel-plate capacitor.

Idealize as 1-dimensional by ignoring the edge effects. 1-d Laplace equation in the vacuum gap (where $\rho = 0$) is

$$\frac{d^2\phi}{dz^2} = 0 \tag{1.45}$$

Solution $\phi = a - Ez$ a,E const.
Hence electric field is $\mathbf{E} = E\hat{\mathbf{z}}$.
Notice how this arises purely from the translational invariance of $\phi\left(\frac{d}{dx} = \frac{d}{dy} = 0\right)$.
Choose $z = 0$ as one plate of capacitor. Other at $z = d$.
Choose $\phi(z = 0) = 0$: reference potential, making $a = 0$.
Potential of other plate: $V = \phi(d) = -Ed$.
The question: how much charge per unit area is there on the plates when the field is E?
Answer by considering a flat elemental volume, with area A surrounding the $+ve$ plate.
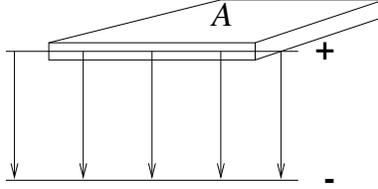
Figure 1.14: Elemental volume for calculating charge/field relationship.

Apply Gauss's law

$$\int_S \mathbf{E}.\mathbf{dS} = \int_A O + (-E)ds = -EA$$
$$= \int_V \frac{\rho}{\epsilon_0} d^3x = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0}\sigma\, A \tag{1.46}$$

Hence

$$\sigma = -\epsilon_0 E. = \epsilon_0 \frac{V}{d} \tag{1.47}$$

Therefore if the total area is A, the total charge Q, and the voltage V between plates are related by

$$Q = \left(\frac{\epsilon_0 A}{d}\right) V \tag{1.48}$$

And the coefficient $\frac{\epsilon_0 A}{d}$ is called capacitance, C. Notice our approach:

- Solve Laplace's equation by choosing coordinates consistent with problem symmetry.

- Obtain charge using Gauss's law to an appropriate trial volume.

1.3.10 Charge on an arbitrary conductor

Consider a conductor, electrostatically charged. Current is zero.
So $\mathbf{E}$ is zero, anywhere <u>inside</u> because of conductivity.
Choose any volume internally: $\mathbf{E} = 0 \Rightarrow \nabla.E = 0 \Rightarrow \rho = 0$. There is <u>no internal charge</u>.
It all resides on <u>surface</u>. At the surface there is an $\mathbf{E}$ just outside. $\mathbf{E}$ is perpendicular to surface $\mathbf{ds}$ because surface is an equipotential (& $\mathbf{E} = -\nabla\phi$). Hence applying Gauss's law to a pill box

$$\int_V \nabla.\mathbf{E}\, d^3x = \int_S \mathbf{E}.\mathbf{dS} = Eds$$
$$= \int \frac{\rho}{\epsilon_0} d^3x = \int \frac{\sigma}{\epsilon_0}ds = \frac{\sigma}{\epsilon_0}ds \tag{1.49}$$

where σ = surface charge density Hence $\sigma = \epsilon_0 E$. Of course, in this general case E ($= E_{\text{normal}}$) is <u>not</u> uniform on the surface but varies from place to place. Again procedure would be: solve ϕ externally from $\nabla^2\phi = 0$; then deduce σ; rather than the other way around.
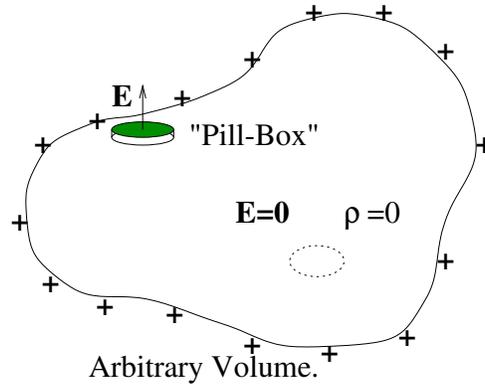
Figure 1.15: Arbitrary-shaped conductor possesses only surface charges related to the local normal field.

1.3.11 Visualizing Electric Potential and Field

$$\mathbf{E} = -\nabla\phi \tag{1.50}$$

Consider a (2-D) contour plot of ϕ. The value of ϕ can be thought of as the potential
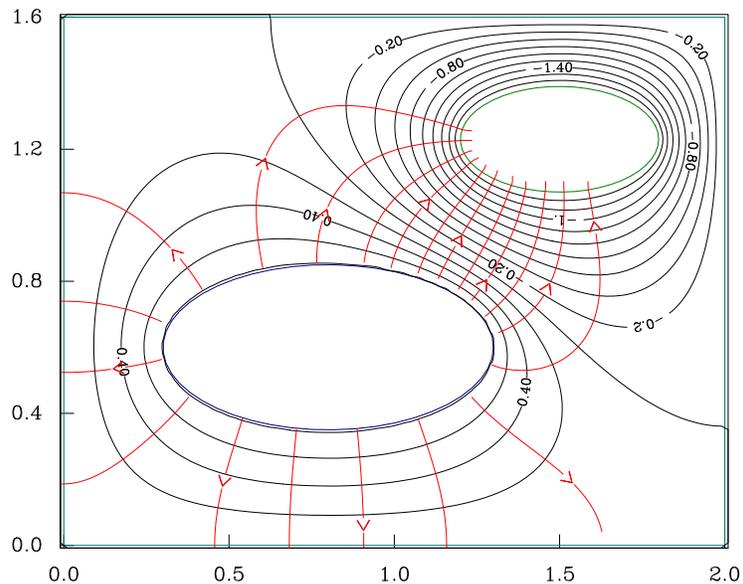
Figure 1.16: Contours of potential and corresponding field-lines (marked with arrows). Only the field-lines emanating from the larger elliptical conductor are drawn.

energy of a charge of 1 Coulomb. Thus there is a perfect analogy to gravitational potential energy and height contours. The force at any point on the hill is downward (on a +*ve* charge), which is perpendicular to the contours of constant ϕ. The strength of the force ($\mathbf{E}$) is proportional to the steepness of the hill: i.e. how close together the <u>contours</u> are (of ϕ). When plotting field-lines, i.e. lines following the electric field direction, we generally also

consider the electric field intensity to be the number of field-lines per unit area. So also the closeness of <u>field-lines</u> indicates field strength. In <u>charge-free</u> regions $\nabla.\mathbf{E} = 0$ implies field-lines have no beginning or end. However if $\rho \neq 0$ then electric field lines <u>do</u> possibly have ends (on the charges). The potential contours never have ends.

1.3.12   Complex Potential Representation 2-D

In charge-free region, $\nabla.\mathbf{E} = 0 \Rightarrow \nabla^2\phi = 0$. This causes there to be an intimate relationship between field-lines and ϕ-contours. In 2 dimensions this relationship allows complex analysis to be used to do powerful analysis of potential problems. Consider a complex function $f(z) = \phi(z) + i\psi(z)$ where $z = x + iy$ is the complex argument with real and imaginary parts x & y; and f has real and imaginary parts ϕ & ψ. f is "analytic" if there exists a well defined complex derivative $\frac{df}{dz}$ (which is also analytic), defined in the usual way as $\lim_{z' \to z} \left(\frac{f(z')-f(z)}{z'-z} \right)$. In order for this limit to be the same no matter what direction (x, y) it is taken in, f must satisfy the "Cauchy-Riemann relations"

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad ; \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{1.51}$$

Which, by substitution imply $\nabla^2\phi = 0$, $\nabla^2\psi = 0$, and also

$$\nabla\phi.\nabla\psi = 0 \tag{1.52}$$

regarding x, y as 2-d coordinates. This shows that

1. The real part of an analytic function solves $\nabla^2\phi = 0$.

2. The contours of the corresponding imaginary part, ψ, then coincide with the electric field-lines.

Finding complex representations of potential problems is one of the most powerful analytic solution techniques. However, for practical calculations, numerical solution techniques are now predominant.

1.4   Electric Current in Distributed Media

Ohms law, $V = IR$, relates voltage current and resistance for a circuit or discrete element. However we often care not just about the total current but about the current <u>density</u> in finite-sized conductors (e.g. electromagnets). This requires a local Ohm's law which is

$$\mathbf{E} = \eta\mathbf{j} \tag{1.53}$$

where η is the medium's electric <u>resistivity</u>. Often the <u>conductivity</u> $\sigma = 1/\eta$ is used. $\mathbf{j} = \sigma\mathbf{E}$, (but I'll try to avoid confusion with surface charge <u>density</u> σ). Such a linear relationship applies in most metals.

1.4.1 Steady State Conduction

Conservation of charge can be written

$$\nabla.\mathbf{j} = -\frac{\partial \rho}{\partial t} \tag{1.54}$$

so, in steady state, $\nabla.\mathbf{j} = 0$, i.e.

$$\nabla.\left(\frac{1}{\eta}\mathbf{E}\right) = (\mathbf{E}.\nabla)\frac{1}{\eta} + \frac{1}{\eta}\nabla.\mathbf{E} = 0 \tag{1.55}$$

If conductivity is uniform ($\nabla\frac{1}{\eta} = 0$) or invariant along $\mathbf{E}$, we therefore have $\nabla.\mathbf{E} = 0 \Rightarrow \rho = 0$. "Uniform conductivity conductors acquire zero volume charge density in steady state".

1.4.2 Conductor Boundary Conditions (Steady Currents)
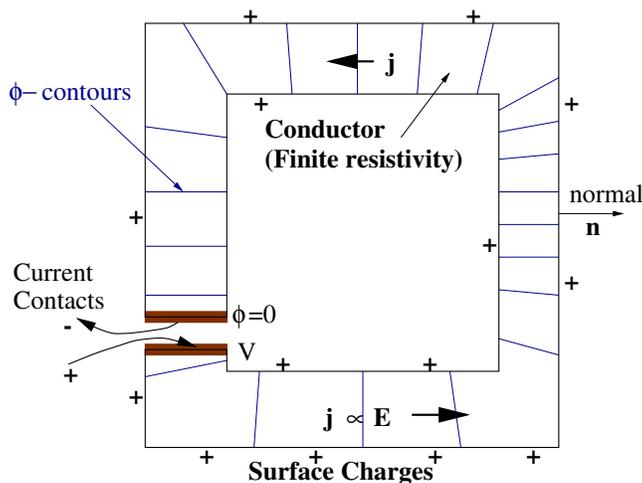
Figure 1.17: A distributed conductor of finite conductivity, carrying current.

If currents are flowing so that $\mathbf{E} \neq 0$ in the conductor then conductors are not equipotential surfaces for solutions of Laplace's equation outside. Surface charges (only) are present on the conductor (uniform η). No current flows through the conductor surface (except at contacts) so

$$\mathbf{j}.\mathbf{n} = 0 \Rightarrow \mathbf{E}.\mathbf{n} = 0 \tag{1.56}$$

inside conductor, while outside we have

$$\mathbf{E}.\mathbf{n} = \epsilon_0\sigma \tag{1.57}$$

surface charge density. Normal components

$$E_n|_{inside} = 0 \; [E_n]^{outside}_{inside} = \sigma/\epsilon_0 \tag{1.58}$$
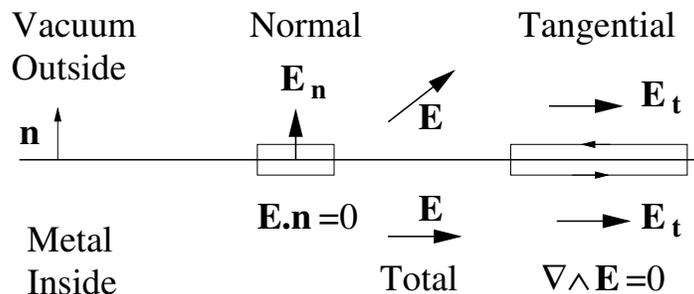
Figure 1.18: Boundary conditions across a conductor/vacuum (or insulator) interface.

Tangential components

$$[E_t]_{inside}^{outside} = 0 \tag{1.59}$$

In particular, for solving $\nabla^2\phi = 0$ inside uniform η conductor, at conductor boundary:

$$\nabla\phi.\mathbf{n} = 0 \qquad \text{(Neumann B.C.)} \tag{1.60}$$

unlike the usual electrostatic B.C. ϕ = given. At electrical contacts ϕ given might be appropriate. A general approach to solving a distributed steady-current problem with uniform-η media:

1. Solve Laplace's equation $\nabla^2\phi = 0$ <u>inside</u> conductors using Dirichlet (ϕ-given) or possibly inhomogeneous Neumann ($\nabla\phi|_n$ = given) BC's at contacts and $\nabla\phi.\mathbf{n} = 0$ at insulating boundaries.

2. Solve Laplace's equation $\nabla^2\phi = 0$ <u>outside</u> conductors using ϕ = given (Dirichlet) B.C. with the ϕ taken from the internal solution.

1.5   Magnetic Potential

Magnetic field has zero divergence $\nabla.\mathbf{B} = 0$. For any vector field[2] $\mathbf{B}$, $\nabla.\mathbf{B} = 0$ is a necessary and sufficient condition that $\mathbf{B}$ can be written as the curl of a vector potential $\mathbf{B} = \nabla \wedge \mathbf{A}$.

1.5.1 $\nabla.\mathbf{B} = 0$ Necessary

$$\nabla.(\nabla \wedge \mathbf{A}) \;=\; \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}^{(z)} - \frac{\partial A_y}{\partial z}^{(y)}\right) \tag{1.61}$$

$$+ \; \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}^{(x)} - \frac{\partial A_z}{\partial x}^{(z)}\right) \tag{1.62}$$

$$+ \; \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}^{(y)} - \frac{\partial A_x}{\partial y}^{(x)}\right) \quad = 0 \tag{1.63}$$

---

[2]satisfying $|\mathbf{B}| \to 0$ as $|\mathbf{x}| \to= \infty$ fast enough

So only divergenceless fields can be represented.

1.5.2 $\nabla.\mathbf{B} = 0$ Sufficient (outline proof by construction)

Consider the quantity

$$\mathbf{K}(\mathbf{x}) = \int \frac{\mathbf{B}(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|} d^3 x' \qquad , \tag{1.64}$$

a vector constructed from the integral of each Cartesian component of $\mathbf{B}$. Applying our knowledge of the Green function solution of Poisson's equation, we know:

$$\nabla^2 \mathbf{K} = -\mathbf{B} \qquad . \tag{1.65}$$

Vector operator theorem (for any $\mathbf{v}$):

$$\nabla \wedge (\nabla \wedge \mathbf{v}) = \nabla (\nabla.\mathbf{v}) - \nabla^2 \mathbf{v} \qquad . \tag{1.66}$$

Hence

$$\mathbf{B} = -\nabla^2 \mathbf{K} = \nabla \wedge (\nabla \wedge \mathbf{K}) - \nabla (\nabla.\mathbf{K}) \tag{1.67}$$

We have proved Helmholtz's theorem that any vector field can be represented as the sum of grad + curl.] When $\nabla.\mathbf{B} = 0$ and $|\mathbf{B}| \to 0$ (fast enough) as $|\mathbf{x}| \to \infty$, one can show that $\nabla.\mathbf{K} = 0$ and so we have constructed the required vector potential

$$\mathbf{A} = \nabla \wedge \mathbf{K} = \nabla \wedge \int \frac{\mathbf{B}(\mathbf{x}')}{4\pi} \frac{d^3 x'}{|\mathbf{x}-\mathbf{x}'|} \qquad . \tag{1.68}$$

Notice that we have constructed $\mathbf{A}$ such that $\nabla.\mathbf{A} = 0$. However $\mathbf{A}$ is undetermined from $\mathbf{B}$ because we can add to it the gradient of an arbitrary scalar without changing $\mathbf{B}$, since $\nabla \wedge \nabla \chi = 0$. So in effect we can make $\nabla.\mathbf{A}$ equal any desired quantity $\psi(\mathbf{x})$ by adding to $\mathbf{A} \nabla \chi$ such that $\nabla^2 \chi = \psi$. Choosing $\nabla.\mathbf{A}$ is known as choosing a "Gauge" $\nabla.\mathbf{A} = 0$ is the "Coulomb Gauge".

1.5.3 General Vector Potential Solution (Magnetostatic)

Static Ampere's law $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$. Now

$$\begin{aligned} \mu_0 \mathbf{j} &= \nabla \wedge (\nabla \wedge \mathbf{A}) \\ &= \nabla (\nabla.\mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A} \qquad \text{(Coulomb Gauge)}. \end{aligned} \tag{1.69}$$

Hence Cartesian components of $\mathbf{A}$ are solutions of Poisson equation

$$\nabla^2 A_i = -\mu_0 j_i \tag{1.70}$$

Using our general solution of Poisson's equation (see eq 1.38):

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|} d^3x'} \tag{1.71}$$

Resulting $\mathbf{B}$:

$$\begin{aligned}
\mathbf{B} &= \nabla \wedge \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla \wedge \frac{\mathbf{j}(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|} d^3x' \\
&= \frac{\mu_0}{4\pi} \int -\mathbf{j}(\mathbf{x'}) \wedge \nabla \frac{1}{|\mathbf{x} - \mathbf{x'}|} d^3x' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(x') \wedge (\mathbf{x} - \mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|^3} d^3x'
\end{aligned} \tag{1.72}$$

This is the distributed-current version of the law of <u>Biot and Savart</u> (dating from $\sim$ 1820). For a wire carrying current I the integral over volume $\mathbf{j}$ is replaced by the integral I $\mathbf{dl}$ i.e.

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi} \int -\frac{(\mathbf{x} - \mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|^3} \wedge \mathbf{j}\, d^3x = \frac{\mu_0}{4\pi} \int -\frac{(\mathbf{x} - \mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|^3} \wedge I\mathbf{dl} \qquad .} \tag{1.73}$$

The Biot-Savart law gives us a direct means to calculate $\mathbf{B}$ by integrating over $\mathbf{j}(\mathbf{x'})$, numerically if necessary. However this integration brute-force method is excessively computationally intensive and if symmetries are present in the problem we can use them to simplify.

1.5.4 Cartesian Translational Symmetry (2-d x, y)
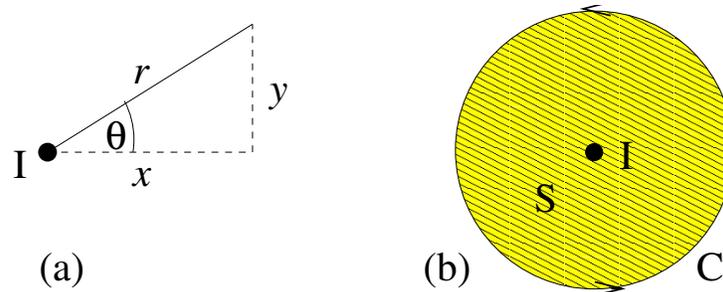
Figure 1.19: (a) The coordinates with respect to an infinite straight filament carrying current I, and (b) the contour and surface for use with Ampere's law.

If we consider a situation where $\partial/\partial z = 0$, corresponding to infinite straight parallel currents in z-direction $\mathbf{j} = j(x, y)\hat{\mathbf{z}}$. Our general vector potential solution shows us immediately that $\mathbf{A} = A\hat{\mathbf{z}}$, $A_x = A_y = 0$. (Assuming $\mathbf{A}, \mathbf{B} \to 0$ at ∞, i.e. no 'external' sources.) That fact tells us that $B_z = (\nabla \wedge \mathbf{A})_z = 0$. We can consider the elementary building block of this problem to be the single infinitesimal filament. Formally $j = I\delta(x)\delta(y)$. We could calculate $\mathbf{B}(\mathbf{x})$ by integrating over this filament. <u>Far</u> easier to use Ampere's Law directly

$$\int_S (\nabla \wedge \mathbf{B}) \cdot \mathbf{ds} = \int_C \mathbf{B} \cdot \mathbf{dl} = \int_S \mu_0 \mathbf{j} \cdot \mathbf{ds} = \mu_0 I \tag{1.74}$$

By symmetry $\oint_C \mathbf{B}.\mathbf{dl} = 2\pi r B_\theta$
So

$$B_\theta = \frac{\mu_0 I}{2\pi r} \quad . \tag{1.75}$$

Also $B_r = 0$ by applying Gauss's Theorem to a volume (of unit length in z-dir)

$$0 = \int_V \nabla.\mathbf{B}\, d^3x = \int_S \mathbf{B}.\mathbf{dS} = 2\pi r B_r \tag{1.76}$$

by symmetry.
Thus Maxwell's equations immediately show us what the 2-d Green function solving $\nabla \wedge \mathbf{B} = \mu_0 I \hat{\mathbf{z}} \delta(\mathbf{x} - \mathbf{x}')$ is

$$\mathbf{B} = \hat{\boldsymbol{\theta}} \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{(x - x')^2 + (y - y')^2}} \tag{1.77}$$

Any general $j(x, y)$ can be handled by 2-d integration using this function.

1.5.5 Cylindrical Symmetry (Circular Loops with common axis)

If there exist cylindrical coordinates (r, θ, z) such that $\partial/\partial\theta = 0$, $\mathbf{j} = j\hat{\theta}$. Then by symmetry $\mathbf{A} = A_\theta \hat{\theta}$, $B_\theta = 0$. This situation turns out to be soluble analytically but only in terms of the special functions known as Elliptic Integrals.   If
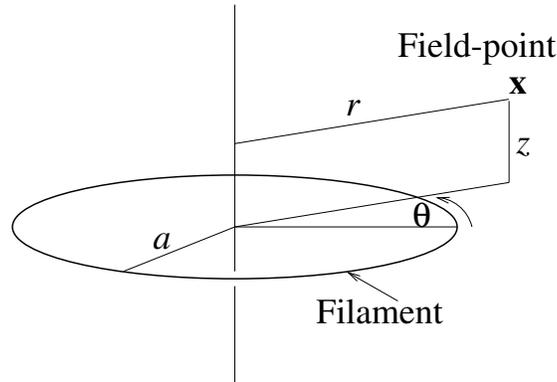
Figure 1.20: Cylindrical Coordinates near a circular current-carrying filament.

$$j_\theta = I\delta\,(r - a)\,\delta\,(z) \tag{1.78}$$

then

$$A_\theta(r, z) = \frac{\mu_0}{4\pi} 2\sqrt{\frac{a}{r}} \left[\frac{(2 - k^2)\,K(k) - 2E(k)}{k} \right] \tag{1.79}$$

where

$$k^2 \equiv \frac{4ra}{(r + a)^2 + z^2} \tag{1.80}$$

and K, E are the complete elliptical integrals of the first and second kind. This general form is so cumbersome that it does not make general analytic calculations tractable but it makes numerical evaluation easier by using canned routines for $K(k)$ & $E(k)$. On axis $(r = 0)$ the field is much simpler

$$\mathbf{B} = B\hat{\mathbf{z}} = \frac{\mu_0 I}{4\pi} \frac{2\pi a^2}{(z^2 + a^2)^{\frac{3}{2}}} \hat{\mathbf{z}} \qquad . \tag{1.81}$$

1.5.6 General Property of Symmetry Situations: Flux Function

When there is a symmetry direction, the component of $\mathbf{B}$ perpendicular to that direction can be expressed in terms of a "flux function". The magnetic flux between two positions is defined as the $\mathbf{B}$-field flux crossing a surface spanning the gap (per unit length if translational). Since $\nabla.\mathbf{B} = 0$ it does not matter how the surface gets from the ref-point to P (provided it
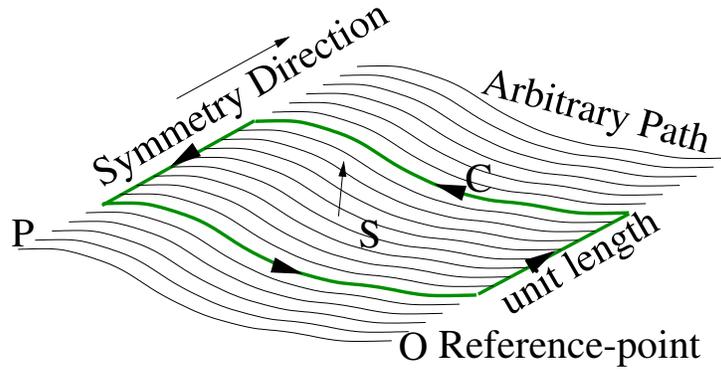
Figure 1.21: Path from a reference point to a field point defines a surface to which Stokes' theorem is applied, in a situation of translational symmetry.

stays symmetric). So the function

$$\psi \equiv \int_S \mathbf{B}.\mathbf{dS} \tag{1.82}$$

is well defined. For translational ($\hat{\mathbf{z}}$) symmetry, a consequence is

$$\mathbf{B}_\perp = -\hat{\mathbf{z}} \wedge \nabla\psi \tag{1.83}$$

This arises because

$$\psi = \int_S \mathbf{B}.\mathbf{dS} = \int \nabla \wedge \mathbf{A}.\mathbf{dS} = A_z(P) - A_z(0) \tag{1.84}$$

So really ψ is identical to the z-component of the vector potential and

$$
\begin{aligned}
\mathbf{B} = \nabla \wedge \mathbf{A} \quad &= \quad B_z\hat{\mathbf{z}} + \nabla \wedge (A_z\hat{\mathbf{z}}) = B_z\hat{\mathbf{z}} + \mathbf{B}_\perp \\
\mathbf{B}_\perp \quad &= \quad A_z\nabla \wedge \hat{\mathbf{z}} + (\nabla A_z) \wedge \hat{\mathbf{z}} \\
&= \quad -\hat{\mathbf{z}} \wedge \nabla A_z = -\hat{\mathbf{z}} \wedge \nabla\psi
\end{aligned}
\tag{1.85}
$$

$\mathbf{B}_\perp$ is the part of the field perpendicular to $\hat{\mathbf{z}}$. There could also be B_z.

For cylindrical symmetry some more variations arise from curvilinear coordinate system. There are even other symmetries, for example helical!

1.6 Electromagnetism and Magnets

1.6.1 Simple Solenoid
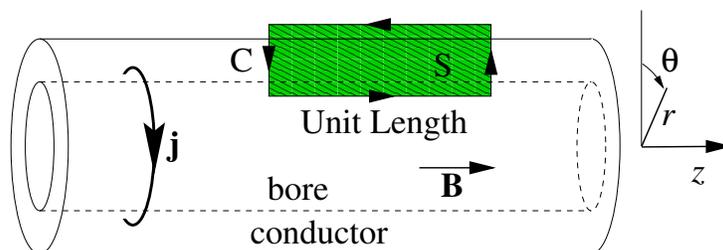
Figure 1.22: Idealized long solenoid magnet coil.

A 'Long' solenoid has a translational symmetry so B is independent of z, as well as of θ. (Except near ends). So

$$0 = \nabla.\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}rB_r + \underbrace{\frac{\partial}{r\partial\theta}B_\theta}_{=0} + \underbrace{\frac{\partial}{\partial z}B_z}_{=0} \tag{1.86}$$

So

$$rB_r = \text{const.} \qquad \text{and hence} \qquad B_r = 0 \quad . \tag{1.87}$$

Also

$$\nabla \wedge \mathbf{B} = \left(\frac{1}{r}\frac{\partial}{\partial r}rB_\theta\right)\hat{\mathbf{z}} + \left(-\frac{\partial}{\partial r}B_z\right)\hat{\boldsymbol{\theta}} = \mu_0\mathbf{j} \qquad \text{(steady)} \tag{1.88}$$

Inside the bore of the magnet, $\mathbf{j} = 0$ so

$$rB_\theta = \text{const.} \qquad \text{and hence} \qquad B_\theta = 0 \quad . \tag{1.89}$$

(actually if $j_z = 0$ everywhere then $B_\theta = 0$ everywhere, as may be seen immediately from the Biot-Savart law). Also

$$\frac{\partial B_z}{\partial r} = 0 \qquad \text{and hence} \qquad B_z = \text{const} \quad . \tag{1.90}$$

Use the surface and bounding curve shown and write

$$\int_S \mu_0\mathbf{j}.\mathbf{dS} = \int_S \nabla \wedge \mathbf{B}.\mathbf{ds} = \oint_C \mathbf{B}.\mathbf{dl} \tag{1.91}$$

So $\mu_0 \times$ current <u>per unit length</u> (denoted J_θ) gives

$$\mu_0 J_\theta = B_{z\ inside} - B_{z\ outside} \tag{1.92}$$

But (by same approach) if $B = 0$ at infinity $B_{z\ outside} = 0$. So, inside

$$B_z = \mu_0 J_\theta \tag{1.93}$$

Profile of field in coil: is determined by the current density in the coil:
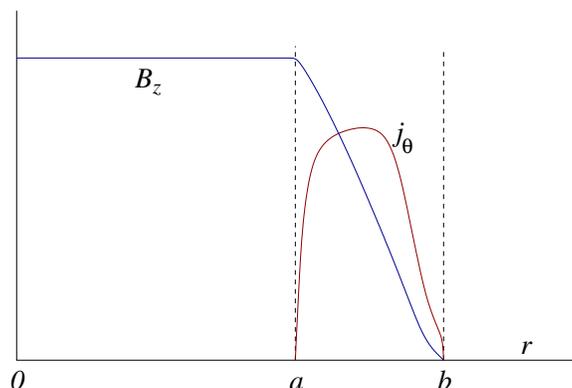
Figure 1.23: The field profile within the conductor region of the coil depends on the current-density profile.

$$\frac{dB_z}{dr} = -\mu_0 j_\theta \tag{1.94}$$

$$B_{zb} - B_{za} = -\int_a^b \mu_0 j_\theta dr = -\mu_0 J_\theta \quad . \tag{1.95}$$

(as before). Notice that all this is independent of coil thickness $(b - a)$. Coils are usually multi-turn so

$$B_z = \mu_0 n I \tag{1.96}$$

where n is turns per unit length, I is current in each turn.

$$J_\theta = n I \qquad . \tag{1.97}$$

1.6.2 Solenoid of Arbitrary Cross-Section

$$\frac{\partial}{\partial z} = 0 \tag{1.98}$$

Consider Biot-Savart Law, expressed as vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \qquad . \tag{1.99}$$

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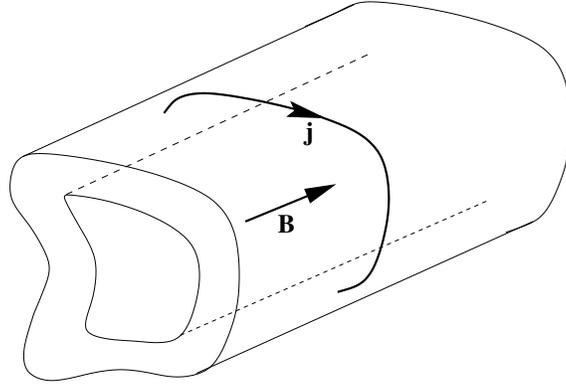
Figure 1.24: Solenoid of arbitrary cross-section.

If currents all flow in azimuthal direction, i.e. $j_z = 0$, then $A_z = 0$.

$$\Rightarrow \qquad B_x = B_y = 0 \qquad \text{(everywhere.)} \tag{1.100}$$

Then integral form of Ampere's law is still

$$B_z(\text{inside}) = \mu_0 J_p \tag{1.101}$$

where J_p is total current in azimuthal direction per unit length.

1.6.3 Coil Types

(a) Wire (Filament):
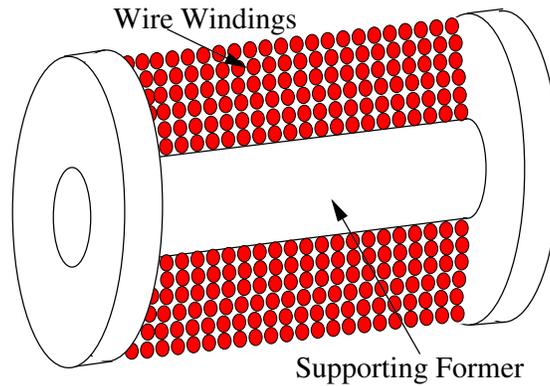
Figure 1.25: Section through a wire-wound magnet coil.

Multiple layers wound on a former. Usually only for low-field low-current work.

(b) Tape wound:

Each coil consists of a spiral-wound tape, n_t turns. Many coils stacked to form a solenoid. Say n_c coils per unit length $n = n_t.n_c$.

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Single Tape-Wound Coil   Stacked Assembly of Coils.
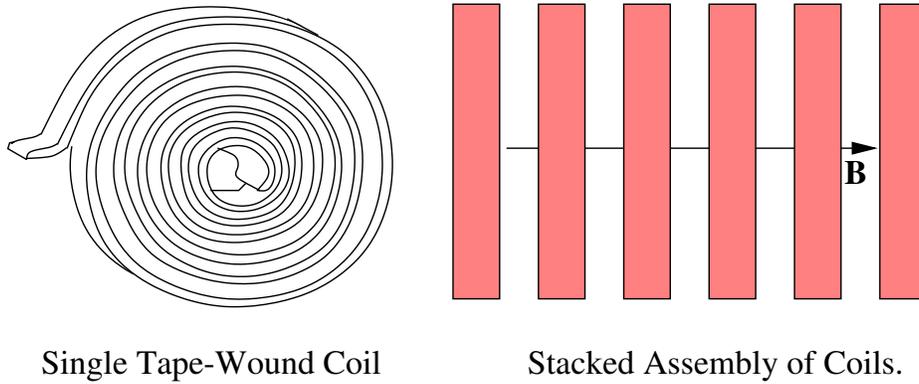
Figure 1.26: Tape-wound coils are stacked to produce a solenoid.

(c) Pancake:

Similar to tape but using square or rectangular conductor. (Fewer turns/coil).

(d) Plate Coils:

n_c turns/unit-length
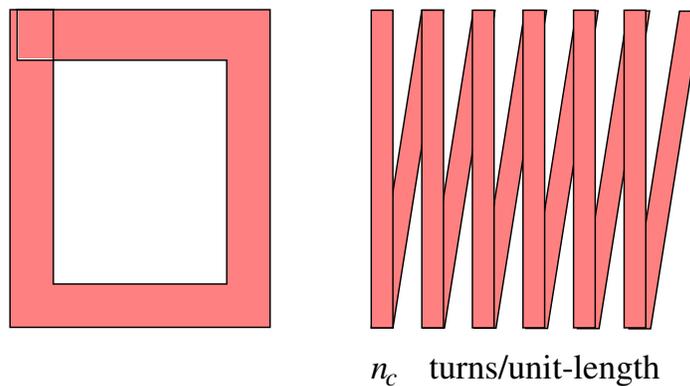
Figure 1.27: A picture-frame type plate coil and the configuration of a solenoid.

Each turn is made of plate. Whole is a single helix (topologically). Plates may be spaced by air or solid insulator gap. $n = n_c$.

There are many other configurations of electromagnet, designed for a tremendous variety of applications. Most require numerical computation to determine the field and its spatial variation.

1.6.4   Magnetic Dipole

The magnetic field from a "localized" current distribution. Suppose we want the field at a point $\mathbf{x}$ which is far from the currents, in the sense that for all points $\mathbf{x}'$ where $\mathbf{j}(\mathbf{x}')$ is
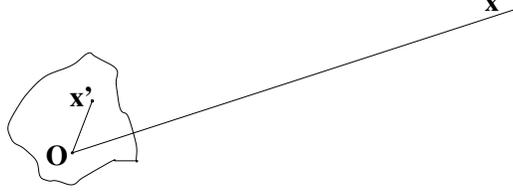
Figure 1.28: Currents localized to a small region close to the origin, with the field point far away.

non-negligible, $|\mathbf{x}'| << |\mathbf{x}|$, (relative to an origin near the currents). The general formula for $\mathbf{A}$:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{1.102}$$

can be approximated by writing

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{(x^2 - 2\mathbf{x}.\mathbf{x}' + x'^2)^{1/2}} \approx \frac{1}{|\mathbf{x}|}\left(1 + \frac{\mathbf{x}.\mathbf{x}'}{|\mathbf{x}|^2} + ...\right) \tag{1.103}$$

so

$$\mathbf{A} \simeq \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\left[\int \mathbf{j}(\mathbf{x}')\, d^3x' + \frac{1}{|\mathbf{x}|^2}\int \mathbf{x}.\mathbf{x}'\mathbf{j}(\mathbf{x}')\, d^3x'\right] \quad . \tag{1.104}$$

Now we convert these integrals into more convenient expressions using $\nabla.\mathbf{j} = 0$. Actually the first one is zero. This follows immediately from the identity

$$\nabla.(\mathbf{jx}) = \mathbf{x}(\nabla.\mathbf{j}) + (\mathbf{j}.\nabla)\mathbf{x} = \mathbf{j} \tag{1.105}$$

(which uses $\nabla\mathbf{x} = \mathbf{I}$ i.e. $\partial x_i/\partial x_j = \delta_{ij}$, and $\nabla.\mathbf{j} = 0$). So

$$\int \mathbf{j}\, d^3x' = \int \nabla'.(\mathbf{jx}')\, d^3x' = \int_S \mathbf{x}'\mathbf{j}.\mathbf{dS} = 0 \quad , \tag{1.106}$$

for any surface S that encloses all currents so that $\mathbf{j} = 0$ on S. An intuitive way to see this is that $\int \mathbf{j}d^3x$ is the average velocity of charges, and must be zero. The second term is simplified using the same identity but being careful to distinguish between $\mathbf{x}$ and $\mathbf{x}'$, and using notation ∇' to denote the gradient operator that operates on $\mathbf{x}'$, $\mathbf{j}(\mathbf{x}')$, <u>not</u> on $\mathbf{x}$.

$$\begin{aligned}
\int (\mathbf{x}.\mathbf{x}')\mathbf{j}(\mathbf{x}')\, d^3x' &= \int (\mathbf{x}.\mathbf{x}')\nabla'.(\mathbf{jx}')\, d^3x' \\
&= \int \nabla'.(\mathbf{jx}'(\mathbf{x}.\mathbf{x}')) - \mathbf{x}'\mathbf{j}.\nabla'(\mathbf{x}'.\mathbf{x})\, d^3x' \\
&= -\int \mathbf{x}'\mathbf{j}.(\nabla'\mathbf{x}').\mathbf{x}\, d^3x' \\
&= -\int \mathbf{x}'\mathbf{j}.\mathbf{I}.\mathbf{x}\, d^3x' = -\int \mathbf{x}'(\mathbf{j}.\mathbf{x})\, d^3x' \tag{1.107}
\end{aligned}$$

But

$$\mathbf{x} \wedge (\mathbf{x}' \wedge \mathbf{j}) = (\mathbf{j}.\mathbf{x})\mathbf{x}' - (\mathbf{x}.\mathbf{x}')\mathbf{j} \quad . \tag{1.108}$$

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So

$$\int \mathbf{x} \wedge (\mathbf{x}' \wedge \mathbf{j}) \, d^3 x' = -2 \int \mathbf{x} . \mathbf{x}' \mathbf{j} \, d^3 x' \quad , \tag{1.109}$$

by the integral relation just proved. [This identity is true for <u>any</u> $\mathbf{x}$]. Therefore our approximation for $\mathbf{A}$ is

$$\mathbf{A} (\mathbf{x}) = -\frac{\mu_0}{4\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3} \wedge \left(\frac{1}{2} \int \mathbf{x}' \wedge \mathbf{j} (\mathbf{x}') \, d^3 x' \right) \tag{1.110}$$

or

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \wedge \mathbf{x}}{|\mathbf{x}|^3} \quad , \tag{1.111}$$

where the <u>Magnetic Dipole Moment</u> of the localized current distribution is

$$\mathbf{m} \equiv \frac{1}{2} \int \mathbf{x}' \wedge \mathbf{j} \, d^3 x' \quad . \tag{1.112}$$

We have derived this expression for an arbitrary j distribution; but if the localized current is a loop current filament,
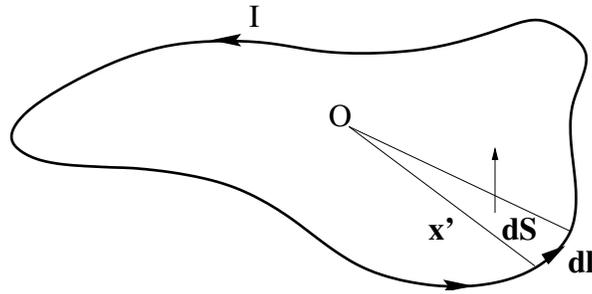
Figure 1.29: Current-carrying loop integration to give dipole moment.

$$\mathbf{m} = \frac{1}{2} \int \mathbf{x}' \wedge \mathbf{j} \, d^3 x' = \frac{1}{2} \int \mathbf{x} \wedge I \mathbf{dl} \quad . \tag{1.113}$$

If the loop is planar,

$$\frac{1}{2} \mathbf{x} \wedge \mathbf{dl} = \mathbf{ds} \quad , \tag{1.114}$$

where $\mathbf{ds}$ is the element of surface. So $\mathbf{m}$ is (current $\times$ area) for a planar filament. The magnetic field is obtained from $\mathbf{B} = \nabla \wedge \mathbf{A}$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left[3 \frac{\mathbf{x}}{|\mathbf{x}|} \left(\frac{\mathbf{x}}{|\mathbf{x}|} . \mathbf{m} \right) - \mathbf{m} \right] \frac{1}{|\mathbf{x}|^3} \tag{1.115}$$

1.6.5 Revisionist History of Electromagnetic Induction

Michael Faraday first showed the effect of induction: a transient current can be induced in one circuit by changes in another. This was $\sim$ 1830. [Faraday knew no mathematics beyond the idea of proportionality EMF $\propto$ rate of change of B-flux]. Suppose history had been different and we knew only the Lorentz force law:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) \tag{1.116}$$

we could have "proved" the necessity of induction by "pure thought".

Assume Galilean Invariance: physical laws must be invariant under changes to moving coordinate systems $x' = x - vt$, $t' = t$. [Universally assumed in Faraday's time. Einstein doesn't come till 1905!] Consider a rigid (wire) circuit moved past a magnet: Each electron
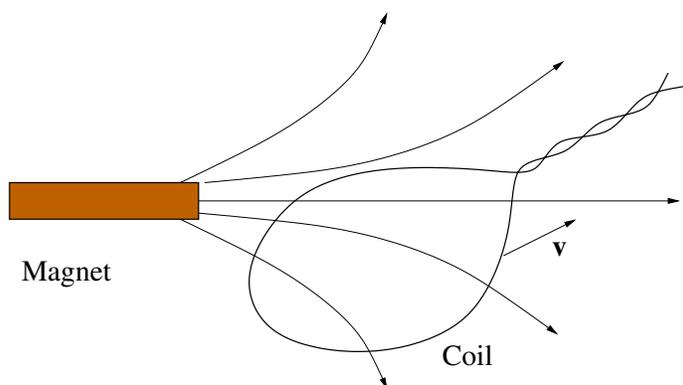
Figure 1.30: A rigid coil moving past a steady magnet.

in the circuit (revisionist!) feels a Lorentz force

$$\mathbf{F} = q\left(\mathbf{v} \wedge \mathbf{B}\right) \tag{1.117}$$

as it is dragged through the magnetic field. The electric field in the rest frame of the magnet is zero. And the total electromotive force (integrated force per unit charge) round the entire circuit is

$$\frac{1}{q}\oint_C \mathbf{F}.\mathbf{dl} = \oint_C \mathbf{v} \wedge \mathbf{B}.\mathbf{dl} \tag{1.118}$$

This is generally a non-zero quantity. In fact, this quantity can be transformed on the basis of purely geometrical considerations. Let's calculate the rate of change of total magnetic flux due to circuit motion, in a static B-field. Apply Gauss' theorem to volume shown in Fig 1.31.

$$
\begin{aligned}
0 &= \int_V \nabla.\mathbf{B}\, d^3x = \int_{S_{total}} \mathbf{B}.\mathbf{dS} = \int_{S'} - \int_S + \int_{ribbon} \mathbf{B}.\mathbf{ds} \\
&= \int_{S'} \mathbf{B}.\mathbf{dS} - \int_S \mathbf{B}.\mathbf{dS} + \int \mathbf{B}.\left(\mathbf{dl} \wedge \mathbf{v}dt\right) \\
&= d\Phi + dt \oint_C \left(\mathbf{v} \wedge \mathbf{B}\right).\mathbf{dl} \tag{1.119}
\end{aligned}
$$
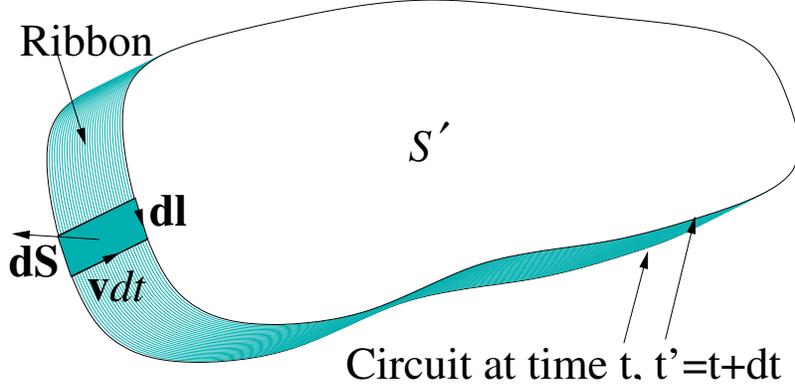
Figure 1.31: Surface elements in the application of Gauss's theorem to succeeding instants of time.

[where $d\Phi$ is change in flux]. So

$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{v} \wedge \mathbf{B}) \,.\mathbf{dl} \quad . \tag{1.120}$$

(pure geometry when $\partial\mathbf{B}/\partial t = 0$).
This equation can alternatively be obtained algebraically by writing

$$\frac{d\Phi}{dt} = \int \frac{d\mathbf{B}}{dt}.\mathbf{dS} = \int (\mathbf{v}.\nabla)\,\mathbf{B}.\mathbf{dS} \tag{1.121}$$

and using

$$\nabla \wedge (\mathbf{B} \wedge \mathbf{v}) = (\mathbf{v}.\nabla)\,\mathbf{B} + (\nabla.\mathbf{v})\,\mathbf{B} - (\mathbf{B}.\nabla)\,\mathbf{v} - (\nabla.\mathbf{B})\,\mathbf{v} = (\mathbf{v}.\nabla)\,\mathbf{B} \quad . \tag{1.122}$$

So

$$\frac{d\Phi}{dt} = \int \nabla \wedge (\mathbf{B} \wedge \mathbf{v})\,.\mathbf{dS} = \oint_C (\mathbf{B} \wedge \mathbf{v})\,.\mathbf{dS} \tag{1.123}$$

Anyway EMF is

$$\frac{1}{q}\oint_C \mathbf{F}.\mathbf{dl} = \oint_C (\mathbf{v} \wedge \mathbf{B})\,.\mathbf{dS} = -\frac{d\Phi}{dt} \tag{1.124}$$

Now we consider the whole situation when the frame of reference is changed to one in which the circuit is stationary and the magnet is moving. By Galilean invariance the total EMF is the same, and

$$\frac{1}{q}\oint \mathbf{F}.\mathbf{dl} = -\frac{d\Phi}{dt} \tag{1.125}$$

But now $\mathbf{v} = 0$, and instead $\mathbf{B}$ is changing so

$$\frac{d\Phi}{dt} = \int \frac{\partial\mathbf{B}}{\partial t}.\mathbf{dS} \quad . \tag{1.126}$$

In this case also the Lorentz force on the charges is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) = q\mathbf{E} \qquad (\text{since } \mathbf{v} = 0) \tag{1.127}$$

There has to be an electric field in this frame of reference. And also

$$\frac{1}{q}\oint \mathbf{F}.\mathbf{dl} = \oint \mathbf{E}.\mathbf{dl} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dS} \tag{1.128}$$

Apply Stokes' theorem to the $\mathbf{E}.\mathbf{dl}$ integral:

$$\int_S \left[\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right].\mathbf{dS} = 0 \quad . \tag{1.129}$$

But this integral has to be zero for all S (and C) which can be true only if its integrand is everywhere zero:

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.130}$$

"Faraday's" Law (expressed in differential form) (which Faraday understood intuitively but could not have formulated in math)

1.6.6   Inductance

Suppose we have a set of circuits with currents $I_i(i = 1...N)$. These are inductively coupled if the current in one gives rise to flux linking the others. Because Ampere's law is linear ($\mathbf{B} \propto \mathbf{j}$), the flux linking circuit j from current I_i is proportional to I_i. Consequently, the total flux linking circuit j can be written

$$\Phi_j = \sum_i M_{ji}I_i \tag{1.131}$$

(Summation over I_i) different currents. M is a matrix. The element M_{ij} is an <u>inductance</u> between currents i and j.. Its units are

$$\frac{flux}{current} \leftrightarrow \frac{\text{Wb}}{\text{A}} \leftrightarrow \text{Henrys} \quad . \tag{1.132}$$

The electromotive force or voltage V_j induced in the j'th circuit is then:

$$V_j = \frac{d}{dt}\Phi_j = \sum_i M_{ji}\dot{I}_i \quad . \tag{1.133}$$

For the simplest case $N = 1$ circuit. $M_{ii} \rightarrow L$ the self inductance

$$V = L\dot{I} \quad . \tag{1.134}$$

It can be shown from Maxwell's equations that M_{ij} is <u>symmetric</u>.

Chapter 2

Particle Motion in Electric and Magnetic Fields

Considering $\mathbf{E}$ and $\mathbf{B}$ to be given, we study the trajectory of particles under the influence of Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) \tag{2.1}$$

2.1   Electric Field Alone

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} \tag{2.2}$$

Orbit depends only on ratio q/m. Uniform $\mathbf{E} \Rightarrow$ uniform acceleration. In one-dimension z, E_z trivial. In multiple dimensions directly analogous to particle moving under influence of gravity. Acceleration gravity $\mathbf{g} \leftrightarrow \frac{q}{m}\mathbf{E}$. Orbits are parabolas. Energy is conserved taking
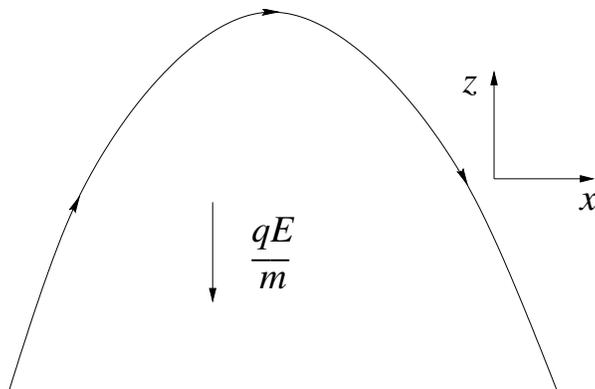
Figure 2.1: In a uniform electric field, orbits are parabolic, analogous to gravity.

into account potential energy

$$P.E. = q\phi \qquad \text{electric potential} \tag{2.3}$$

[Proof if needed, regardless of **E** spatial variation,

$$m\frac{d\mathbf{v}}{dt}.\mathbf{v} = -q\nabla\phi.\mathbf{v} = -q\frac{d\phi}{dt} \tag{2.4}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\frac{d}{dt}(q\phi) \tag{2.5}$$

i.e. $\frac{1}{2}mv^2 + q\phi = \text{const.}$]

A particle gains kinetic energy $q\phi$ when falling through a potential drop $-\phi$. So consider the acceleration and subsequent analysis of particles electrostatically: How much deflection
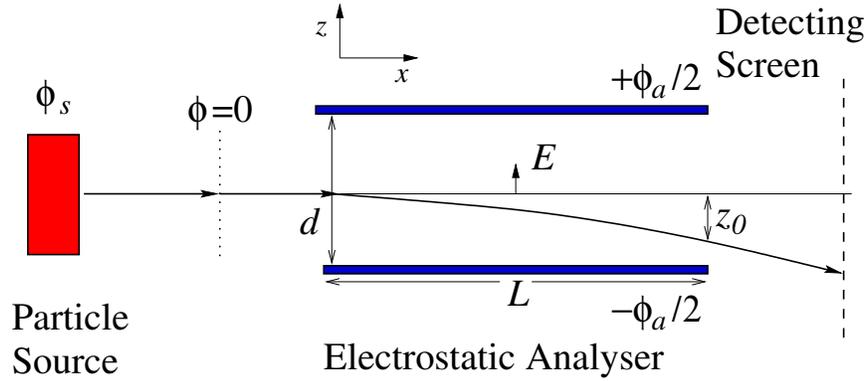
Figure 2.2: Schematic of electrostatic acceleration and analysis.

will there be? After acceleration stage $KE = \frac{1}{2}mv_x^2 = q\phi_s$

$$v_x = \sqrt{\frac{2q\phi_s}{m}} \quad . \tag{2.6}$$

Supposing $\mathbf{E}_a$, field of analyser, to be purely $\hat{\mathbf{z}}$, this velocity is subsequently constant. Within the analyser

$$m\frac{dv_z}{dt} = qE_a \Rightarrow v_z = \frac{q}{m}E_a t = \frac{q}{m}E_a\frac{x}{v_x} \quad . \tag{2.7}$$

So

$$z = \int v_z dt = \frac{q}{m}E_a\frac{t^2}{2} = \frac{q}{m}E_a\frac{1}{2}\frac{x^2}{v_x^2} \quad . \tag{2.8}$$

Hence height at output of analyser is

$$\begin{aligned}
z_o &= \frac{q}{m}E_a\frac{1}{2}\frac{L^2}{v_x^2} = \frac{q}{m}E_a\frac{1}{2}L^2\frac{m}{2q\phi_s} \\
&= -\frac{1}{4}\frac{E_a}{\phi_s}L^2 = +\frac{1}{4}\frac{\phi_a}{\phi_s}\frac{L^2}{d} \tag{2.9}
\end{aligned}$$

using $E_a = -\phi_a/d$. Notice this is independent of q and m! We could see this directly by eliminating the time from our fundamental equations noting

$$\frac{d}{dt} = v\frac{d}{d\ell} \, (= \mathbf{v}.\nabla) \qquad \text{with} \qquad v = \sqrt{\frac{2q(\phi - \phi_s)}{m}} \qquad \text{or} \qquad v = \sqrt{\frac{2q}{m}\left(\phi - \phi_s + \frac{\mathcal{E}_s}{q}\right)}$$

(2.10)

if there is initial energy $\mathcal{E}_s$. So equation of motion is

$$\frac{m}{q}\sqrt{\frac{2q(\phi_s - \phi)}{m}}\frac{d}{d\ell}\left(\sqrt{\frac{2q(\phi_s - \phi)}{m}}\frac{d}{d\ell}\mathbf{x}\right) = 2\sqrt{\phi_s - \phi}\frac{d}{d\ell}\sqrt{\phi_s - \phi}\frac{d\mathbf{x}}{d\ell} = E_a = -\nabla\phi \quad , \quad (2.11)$$

which is independent of q and m. Trajectory of particle in purely electrostatic field depends only on the field (and initial particle kinetic energy/q). If initial energy is zero, can't deduce anything about q, m.

2.2 Electrostatic Acceleration and Focussing

Accelerated charged particle beams are widely used in science and in everyday applications.

Examples:

X-ray generation from e-beams (Medical, Industrial)
Electron microscopes
Welding. (e-beam)
Surface ion implantation
Nuclear activation (ion-beams)
Neutron generation
Television and (CRT) Monitors (fading!)
For applications requiring $\lesssim$ few hundred keV energy electrostatic acceleration is easiest, widest used. Schematically
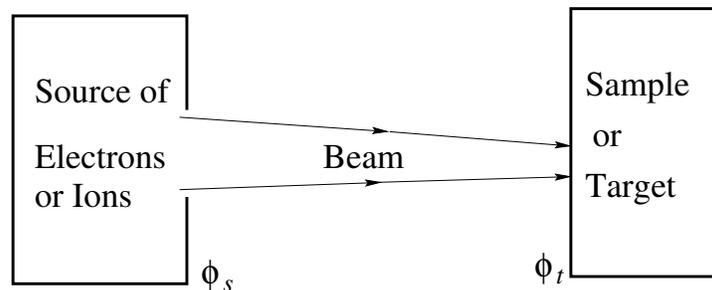
Figure 2.3: Obtaining defined energy from electrostatic acceleration is straightforward in principle. Beam focussing and transport to the target is crucial.

Clearly getting the required energy is simple. Ensure the potential difference is right and particles are singly charged: Energy $(eV) \leftrightarrow$ Potential V. More interesting question: How to focus the beam? What do we mean by focussing?
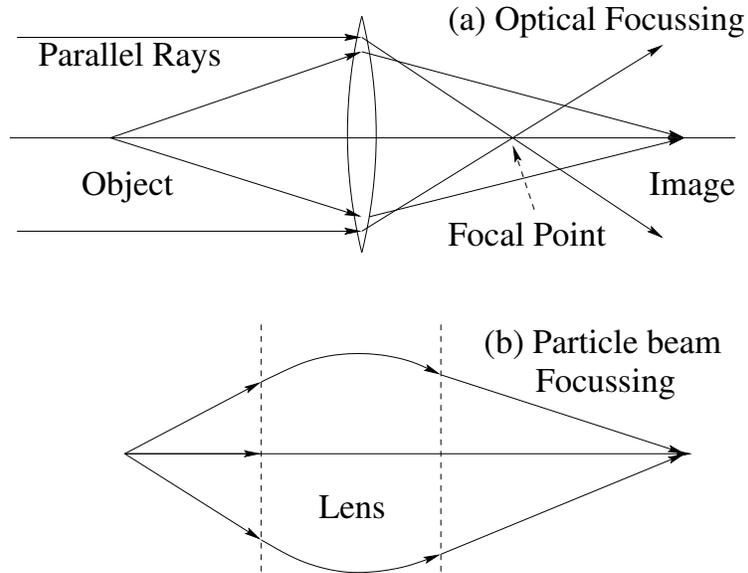
Figure 2.4: Analogy between optical and particle-beam focussing.

What is required of the "Lens"? To focus at a single spot we require the ray (particle path) deviation from a "thin" lens to be systematic. Specifically, all initially parallel rays converge to a point if the lens deviates their direction by θ such that
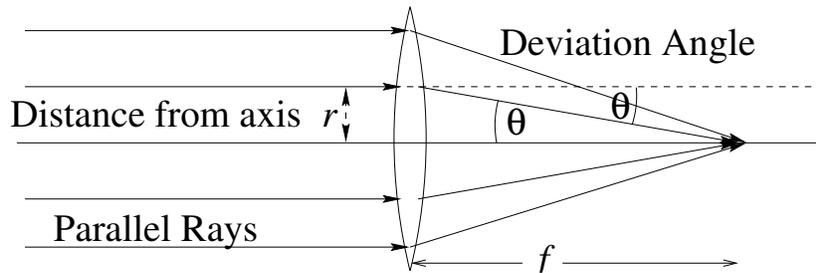
Figure 2.5: Requirement for focussing is that the angular deviation of the path should be a linear function of the distance from the axis.

$$r = f \tan \theta \tag{2.12}$$

and for small angles, θ, $r = f\theta$. This <u>linear</u> dependence $(\theta = r/f)$ of the deviation θ, on distance from the axis, r, is the key property. Electrostatic Lens would like to have (e.g.)

$$E_r = \frac{E_a}{a} r \tag{2.13}$$

but the lens can't have charged solids in its middle because the beams must pass through so (initially) $\rho = 0 \Rightarrow \nabla.\mathbf{E} = 0$. Consequently <u>pure E_r is impossible</u> $(0 = \nabla.\mathbf{E} = \frac{1}{r}\partial(rE_r)/\partial r = 2E_a/a \Rightarrow E_a = 0)$. For an axisymmetric lens $(\partial/\partial\theta = 0)$ we must have both E_r and E_z. Perhaps the simplest way to arrange appropriate E_r is to have an <u>aperture</u> between two
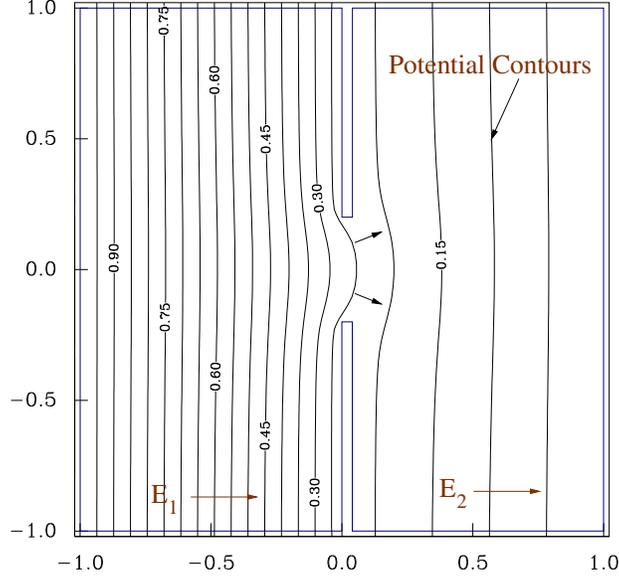
Figure 2.6: Potential variation near an aperture between two regions of different electric field gives rise to focussing.

regions of unequal electric field. The potential contours "bow out" toward the lower field region: giving E_r.

Calculating focal length of aperture Radial acceleration.

$$\frac{dv_r}{dt} = \frac{q}{m}E_r \tag{2.14}$$

So

$$\frac{dv_r}{dz} = \frac{1}{v_z}\frac{dv_r}{dt} = \frac{q}{m}\frac{E_r}{v_z} \tag{2.15}$$

But

$$\nabla.\mathbf{E} = 0 \Rightarrow \frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \frac{\partial E_z}{\partial z} = 0 \tag{2.16}$$

Near the axis, only the linear part of E_r is important i.e.

$$E_r(r,z) \simeq r\left.\frac{\partial E_r}{\partial r}\right|_{r=0} \tag{2.17}$$

So

$$\frac{1}{r}\frac{\partial}{\partial r}rE_r \simeq 2\left.\frac{\partial E_r}{\partial r}\right|_{r=0} \tag{2.18}$$

39
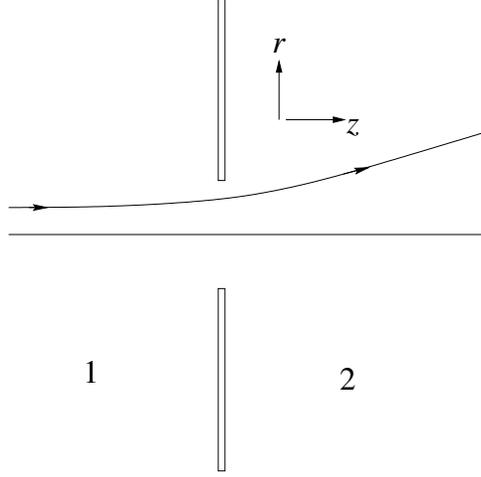
Figure 2.7: Coordinates near an aperture.

and thus

$$2 \left. \frac{\partial E_r}{\partial r} \right|_{r=0} + \frac{\partial E_z}{\partial z} = 0 \tag{2.19}$$

and we may write $E_r \simeq -\frac{1}{2} r \partial E_z / \partial z$. Then

$$\frac{dv_r}{dz} = -\frac{qr}{2mv_z} \frac{\partial E_z}{\partial z} \quad , \tag{2.20}$$

which can be integrated approximately assuming that variations in r and v_z can be neglected in lens to get

$$\delta v_r = [v_r]_{initial}^{final} = \frac{-qr}{2mv_z} [E_z]_1^2 \tag{2.21}$$

The angular deviation is therefore

$$\theta = \frac{-\delta v_r}{v_z} = \frac{+qr}{2mv_z^2} [E_{z2} - E_{z1}] \tag{2.22}$$

and the focal length is $f = r/\theta$

$$f = \frac{2mv_z^2}{q (E_{z2} - E_{z1})} = \frac{4\mathcal{E}}{q (E_{z2} - E_{z1})} \tag{2.23}$$

When E_1 is an accelerating region and E_2 is zero or small the lens is diverging. This means that just depending on an extractor electrode to form an ion beam will give a diverging beam. Need to do more focussing down stream: more electrodes.

2.2.1 Immersion Lens

Two tubes at different potential separated by gap In this case the gap region can be thought of as an aperture but with the electric fields E_1, E_2 the same (zero) on both sides. Previous effect is zero. However two other effects, neglected previously, give focussing:
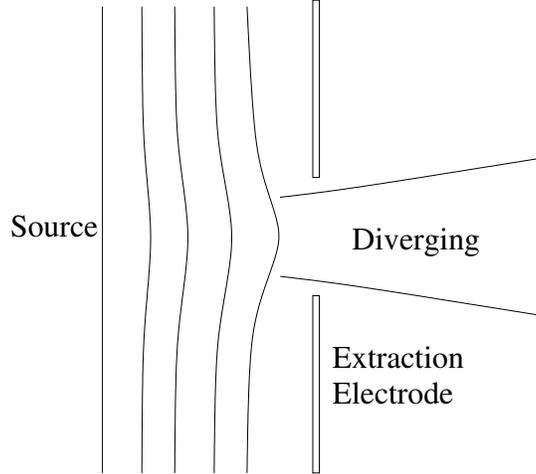
Figure 2.8: The extraction electrode alone always gives a diverging beam.
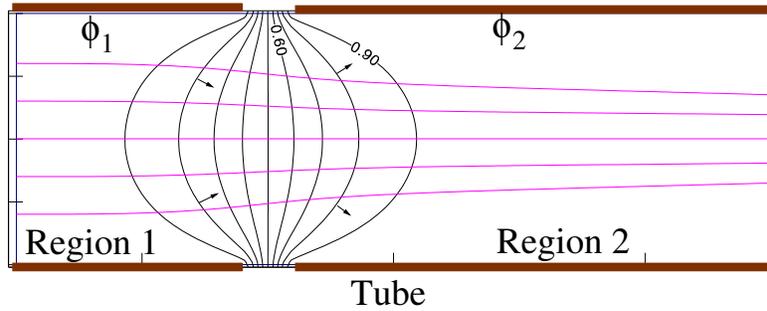
Figure 2.9: An Immersion Lens consists of adjacent sections of tube at different potentials.

1. v_z is not constant.

2. r is not constant.

Consider an accelerating gap: $q(\phi_2 - \phi_1) < 0$.

Effect (1)   ions are converged in region 1, diverged in region 2. However because of z-acceleration, v_z is higher in region 2. The diverging action lasts a shorter time. Hence overall <u>converging</u>.

Effect (2)   The electric field E_r is weaker at smaller r. Because of deviation, r is smaller in diverging region. Hence overall <u>converging</u>.

   For a decelerating gap you can easily convince yourself that both effects are still <u>converging</u>. [Time reversal symmetry requires this.] One can estimate the focal length as

$$\frac{1}{f} \simeq \frac{3}{16} \frac{q^2}{\mathcal{E}^2} \int \left(\frac{\partial \phi}{\partial z} \right)^2 \bigg|_{r=0} dz \quad \text{(for weak focussing)} \tag{2.24}$$

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but numerical calculations give the values in figure 2.10 where $\phi_1 = \mathcal{E}/q$. Here $\mathcal{E}$ is the energy in region 1. Effect (2) above, that the focussing or defocussing deviation is weaker
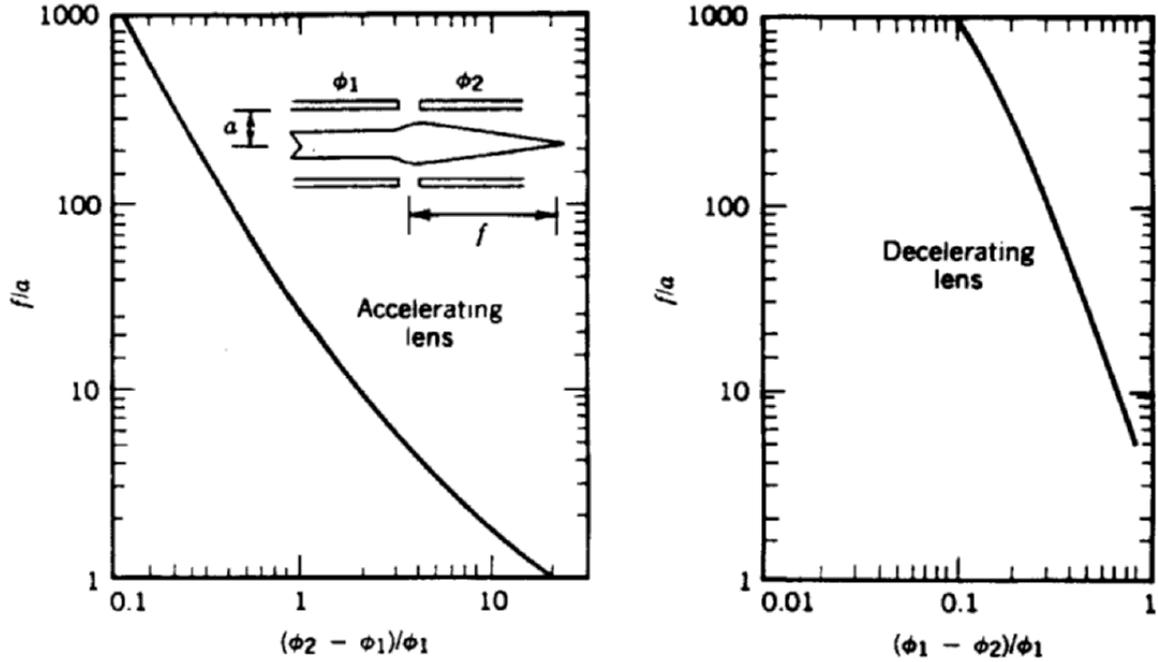
Figure 2.10: Focal length of Electrostatic Immersion Lenses. Dependence on energy per unit charge (ϕ) in the two regions, from S.Humphries 1986

at points closer to the axis, means that it is a general principle that alternating lenses of equal converging and diverging power give a net converging effect. This principle can be considered to be the basis for

2.2.2 Alternating Gradient Focussing

Idea is to abandon the cylindrically symmetric geometry so as to obtain stronger focussing. Consider an electrostatic configuration with $E_z = 0$ and

$$E_x = \frac{dE_x}{dx}x \quad \text{with} \quad \frac{dE_x}{dx} = \text{const.} \tag{2.25}$$

Since $\nabla.\mathbf{E} = 0$, we must have

$$\frac{dE_x}{dx} + \frac{dE_y}{dx} = 0 \Rightarrow \frac{dE_y}{dy} = \text{const} \quad \Rightarrow E_y = -\frac{dE_x}{dx}y \tag{2.26}$$

This situation arises from a potential

$$\phi = \left(x^2 - y^2\right)\left(\frac{1}{2}\frac{dE_x}{dx}\right) \tag{2.27}$$

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so equipotentials are hyperbolas $x^2 - y^2 = const$. If qdE_x/dx is negative, then this field is converging in the x-direction, but $dE_y/dy = -dE_x/dx$, so it is, at the same time, diverging in the y-direction. By using alternating sections of +ve and -ve dE_x/dx a net converging focus can be obtained in <u>both</u> the x and y directions. This alternating gradient approach is very important for high energy particle accelerators, but generally magnetic, not electrostatic, fields are used. So we'll go into it more later.

2.3   Uniform Magnetic field

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{v} \wedge \mathbf{B}\right) \tag{2.28}$$

Take $\mathbf{B}$ in $\hat{\mathbf{z}}$-direction. Never any force in $\hat{z}$-dir. $\Rightarrow v_z = $ constant. Perpendicular dynamics are separate.
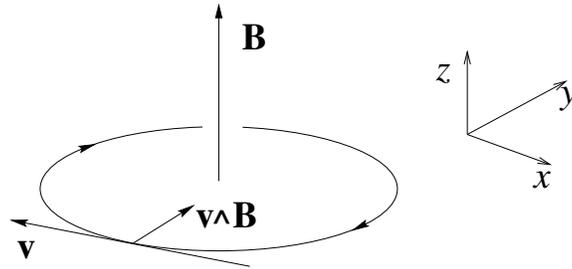
Figure 2.11: Orbit of a particle in a uniform magnetic field.

2.3.1   Brute force solution:

$$\dot{v}_x = \frac{q}{m}v_y B \qquad \dot{v}_y = \frac{-q}{m}v_x B \tag{2.29}$$

$$\Rightarrow \ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB}{m}\right)^2 v_y \tag{2.30}$$

Solution

$$
\begin{aligned}
v_x &= v \sin \frac{qB}{m}t & v_y &= v \cos \frac{qB}{m}t \\
x &= -v\frac{m}{qB}\cos \frac{qB}{m}t + x_0 & y &= v\frac{m}{qB}\sin \frac{qB}{m}t + y_0
\end{aligned}
\quad , \tag{2.31}
$$

the equation of a circle. Center (x_0, y_0) and radius (vm/qB) are determined by initial conditions.

2.3.2 'Physics' Solution

1. Magnetic field force does no work on particle because $\mathbf{F} \perp \mathbf{v}$. Consequently total $|v|$ is constant.

2. Force is thus constant, $\perp$ to $\mathbf{v}$. Gives rise to a circular orbit.

3. Centripetal acceleration gives $\frac{v^2}{r} = \frac{Force}{mass} = q\frac{vB}{m}$ i.e. $r = \frac{mv}{qB}$. This radius is called the Larmor (or gyro) Radius.

4. Frequency of rotation $\frac{v}{r} = \frac{qB}{m} \equiv \Omega$ is called the "Cyclotron" frequency (angular frequency, s^{-1}, not cycles/sec, Hz).

When we add the constant v_z we get a helical orbit. Cyclotron frequency $\Omega = qB/m$ depends only on particle character q, m and B-strength not v (non relativistically, see aside). Larmor Radius $r = mv/qB$ depends on particle momentum $m\mathbf{v}$. All (non-relativistic) particles with same q/m have same Ω. Different energy particles have different r. This variation can be used to make momentum spectrometers.

2.3.3 Relativistic Aside

Relativistic dynamics can be written

$$\frac{d}{dt}\mathbf{p} = q\left([\mathbf{E}+]\,\mathbf{v} \wedge \mathbf{B}\right) \tag{2.32}$$

where relativistic momentum is

$$\mathbf{p} = m\mathbf{v} = \frac{m_0\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad . \tag{2.33}$$

Mass m is increased by factor

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{2.34}$$

relative to rest mass m_0. Since for $\mathbf{E} = 0$ the velocity $|v| = $ const, γ is also constant, and so is m. Therefore dynamics of a particle in a purely magnetic field can be calculated as if it were non-relativistic: $m\,d\mathbf{v}/dt = q(\mathbf{v} \wedge \mathbf{B})$, except that the particle has mass greater by factor γ than its rest mass.

2.3.4 Momentum Spectrometers

Particles passing vertically through slit take different paths depending on mv/q. By measuring where a particle hits the detection plane we measure its momentum/q :

$$2\frac{mv}{qB} = x \qquad : \qquad \frac{mv}{q} = \frac{Bx}{2} \qquad . \tag{2.35}$$
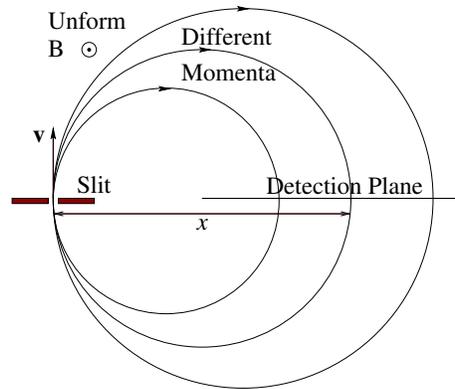
Figure 2.12: Different momentum particles strike the detection plane at different positions.

Why make the detection plane a diameter? Because detection position is least sensitive to velocity direction. This is a form of magnetic focussing. Of course we don't need to make the full 360°, so analyser can be reduced in size.
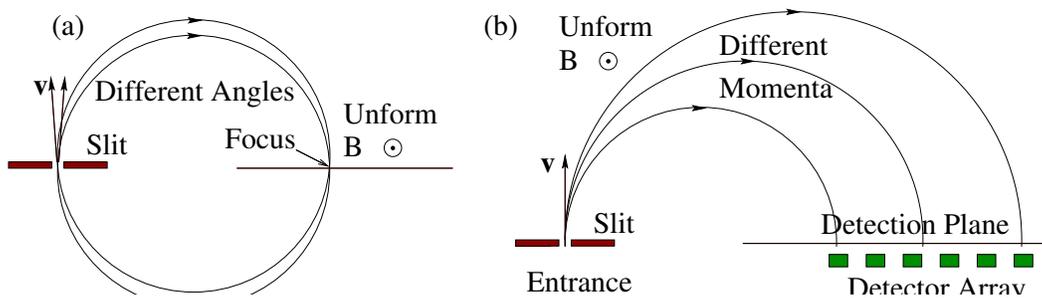
Figure 2.13: (a) Focussing is obtained for different input angles by using 180 degrees of orbit. (b) The other half of the orbit is redundant.

Even so, it may be inconvenient to produce uniform B of sufficient intensity over sufficiently large area if particle momentum is large.

2.3.5 Historical Day Dream (J.J. Thomson 1897)

"Cathode rays": how to tell their charge and mass?

Electrostatic Deflection

Tells only their $energy/q = \mathcal{E}/q$ and we have no independent way to measure $\mathcal{E}$ since the same quantity $\mathcal{E}/q$ just equals accelerating potential, which is the thing we measure.

Magnetic Deflection

The radius of curvature is

$$r = \frac{mv}{qB} \qquad (2.36)$$

So combination of electrostatic and electromagnetic gives us

$$\frac{\frac{1}{2}mv^2}{q} = M_1 \quad and \quad \frac{mv}{q} = M_2 \qquad (2.37)$$

Hence

$$\frac{2M_1}{M_2^2} = \frac{q}{m}. \qquad (2.38)$$

We can measure the charge/mass ratio. In order to complete the job an independent measure of q (or m) was needed. Millikan (1911-13). [Actually Townsend in J.J. Thomson's lab had an experiment to measure q which was within $\sim$ factor 2 correct.]

2.3.6   Practical Spectrometer

In fusion research fast ion spectrum is often obtained by simultaneous electrostatic and electromagnetic analysis E parallel to B. This allows determination of $\mathcal{E}/q$ and q/m $\Rightarrow$ velocity of particle $[\mathcal{E} = \frac{1}{2}mv^2]$.   Thus e.g. deuterons and protons can be distinguished.
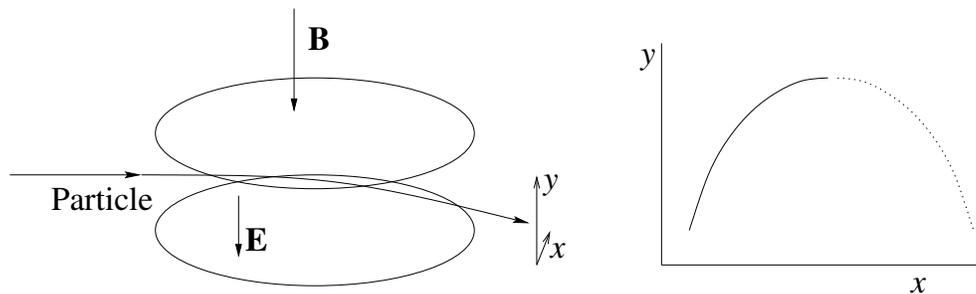
Figure 2.14: E parallel to B analyser produces parabolic output locus as a function of input velocity. The loci are different for different q/m.

However, He^4 and D^2 have the same $\frac{q}{m}$ so one can't distinguish their spectra on the basis of ion orbits.

2.4   Dynamic Accelerators

In addition to the electrostatic accelerators, there are several different types of accelerators based on time-varying fields. With the exception of the Betatron, these are all based on the general principle of arranging for a resonance between the particle and the oscillating fields such that energy is continually given to the particle. Simple example
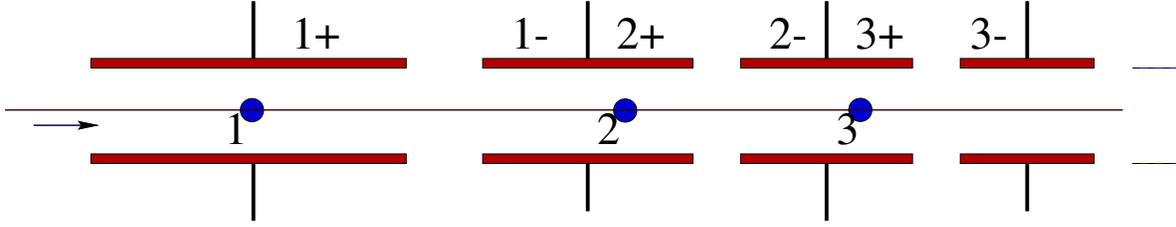
Figure 2.15: Sequence of dynamically varying electrode potentials produces continuous acceleration. Values at 3 times are indicated.

Particle is accelerated through sequence of electrodes 3 at times (1) (2) (3). The potential of electrode is raised from negative to $+ve$ while particle is inside electrode. So at each gap it sees an accelerating E_z. Can be thought of as a successive moving potential hill:
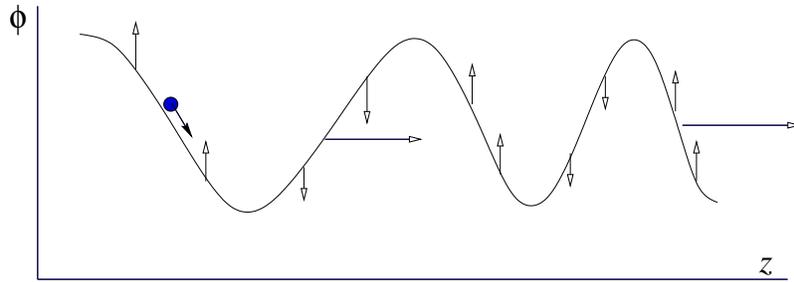
Figure 2.16: Oscillating potentials give rise to a propagating wave.

"Wave" of potential propagates at same speed as particle so it is continuously accelerated. Historically earliest widespread accelerator based on this principle was the cyclotron.

2.4.1 Cyclotron

Take advantage of the orbit frequency in a uniform B-field $\Omega = \frac{qB}{m}$. Apply oscillating potential to electric poles, at this frequency. Each time particle crosses the gap (twice/turn) it sees an accelerating electric field. Resonant frequency

$$f = \frac{\Omega}{2\pi} = \frac{qB}{m2\pi} = 1.52 \times 10^7 B \quad \text{Hz} \tag{2.39}$$

15.2 MHz/T for protons. If magnet radius is R particle leaves accelerator when its Larmor radius is equal to R

$$\frac{mv}{qB} = R \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}\frac{q^2}{m}B^2R^2 \tag{2.40}$$

If iron is used for magnetic pole pieces then $B \lesssim 2T$ (where it saturates). Hence larger accelerator is required for higher energy $\mathcal{E} \propto R^2$. [But stored energy in magnet $\propto R^2 \to R^3$].
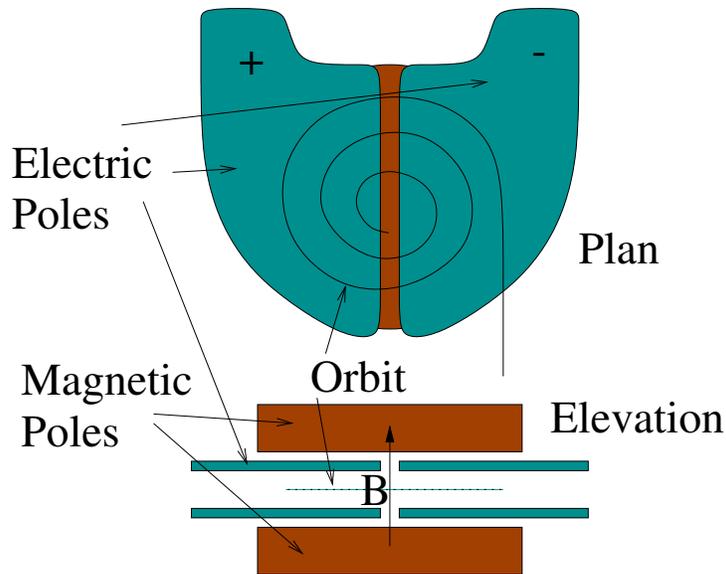
Figure 2.17: Schematic of a Cyclotron accelerator.

2.4.2 Limitations of Cyclotron Acceleration: Relativity

Mass increase $\propto (1 - v^2/c^2)^{-\frac{1}{2}}$ breaks resonance, restricting maximum energy to $\sim$25MeV (protons). Improvement: sweep oscillator frequency (downward). "Synchrocyclotron" allowed energy up to $\sim$500MeV but reduced flux. Alternatively: Increase B with radius. Leads to orbit divergence parallel to B. Compensate with azimuthally varying field for focussing AVF-cyclotron. Advantage continuous beam.

2.4.3 Synchrotron

Vary both frequency and field in time to keep beam in resonance at constant radius. High energy physics (to 800 GeV).

2.4.4 Linear Accelerators

Avoid limitations of electron synchrotron radiation. Come in 2 main types. (1) Induction (2) RF (linacs) with different pros and cons. (RF for highest energy electrons). Electron acceleration: $v = c$ different problems from ion.

2.5 Magnetic Quadrupole Focussing (Alternating Gradient)

Magnetic focussing is preferred at high particle energy. Why? Its force is stronger.
Magnetic force on a relativistic particle qcB.

Electric force on a relativistic particle qE.

E.g. $B = 2T \Rightarrow cB = 6 \times 10^8$ same force as an electric field of magnitude $6 \times 10^8 V/m = 0.6$MV/mm! However magnetic force is perpendicular to $\mathbf{B}$ so an axisymmetric lens would like to have purely azimuthal B field $\mathbf{B} = \hat{\theta}B_\theta$. However this would require a current right
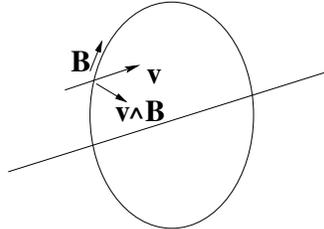
Figure 2.18: Impossible ideal for magnetic focussing: purely azimuthal magnetic field.

where the beam is:

$$\oint \mathbf{B}.\mathbf{d}\ell = \mu_o I. \tag{2.41}$$

Axisymmetric magnetic lens is impossible. However we can focus in one cartesian direction (x, y) at a time. Then use the fact that successive combined focus-defocus has a net focus.

2.5.1 Preliminary Mathematics

Consider $\frac{\partial}{\partial z} = 0$ purely transverse field (approx) B_x, B_y. This can be represented by $\mathbf{B} = \nabla \wedge \mathbf{A}$ with $\mathbf{A} = \hat{\mathbf{z}}A$ so $\nabla \wedge \mathbf{A} = \nabla \wedge (\hat{\mathbf{z}}A) = -\hat{\mathbf{z}} \wedge \nabla A$ (since $\nabla \hat{\mathbf{z}} = 0$). In the vacuum region $\mathbf{j} = 0$ (no current) so

$$0 = \nabla \wedge \mathbf{B} = \nabla \wedge (-\hat{\mathbf{z}} \wedge \nabla A) = -\hat{\mathbf{z}}\nabla^2 A + \underbrace{(\hat{\mathbf{z}}.\nabla)}_{=0} \nabla A \tag{2.42}$$

i.e. $\nabla^2 A = 0$. A satisfies Laplace's equation. Notice then that solutions of electrostatic problems, $\nabla^2 \phi = 0$ are also solutions of (2-d) vacuum magnetostatic problems. The same solution techniques work.

2.5.2 Multipole Expansion

Potential can be expanded about some point in space in a kind of Taylor expansion. Choose origin at point of expansion and use coordinates (r, θ), $x = r\cos\theta$, $y = r\sin\theta$.

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} = 0 \tag{2.43}$$

Look for solutions in the form $A = u(r).w(\theta)$. These require

$$\frac{d^2 w}{d\theta^2} = -\text{const.} \times w \tag{2.44}$$

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and

$$r\frac{d}{dr}r\frac{du}{dr} = \text{const.} \times u. \tag{2.45}$$

Hence w solutions are sines and cosines

$$w = \cos n\theta \qquad \text{or} \qquad \sin n\theta \tag{2.46}$$

where n^2 is the constant in the previous equation and n integral to satisfy periodicity. Correspondingly

$$u = r^n \qquad \text{or} \qquad \ln r \quad, \quad r^{-n} \tag{2.47}$$

These solutions are called "cylindrical harmonics" or (cylindrical) multipoles:

$$\begin{array}{ll} 1 & \ln r \\ r^n \cos n\theta & r^{-n} \cos n\theta \\ r^n \sin n\theta & r^{-n} \sin n\theta \end{array} \tag{2.48}$$

If our point of expansion has no source at it (no current) then the right-hand column is ruled out because no singularity at $r = 0$ is permitted. The remaining multipoles are

$$\begin{array}{ll} 1 & \text{constant irrelevant to a potential} \\ r\cos\theta(=x) & \text{uniform field, } \nabla A \propto \hat{\mathbf{x}} \\ r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2 & \text{non-uniform field} \\ \text{Higher orders} & \text{neglected.} \end{array}$$

The second order solution, $x^2 - y^2$ is called a "quadrupole" field (although this is something of a misnomer). [Similarly $r^3 \cos 3\theta \rightarrow$ "hexapole", $r^4 \cos\theta$ "octupole".] We already dealt with this potential in the electric case.

$$\nabla A = \nabla\left(x^2 - y^2\right) = 2x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} \qquad . \tag{2.49}$$

So

$$-\hat{\mathbf{z}} \wedge \nabla A = -2x\hat{\mathbf{y}} - 2y\hat{\mathbf{x}} \tag{2.50}$$

Force on longitudinally moving charge:

$$\begin{aligned} \mathbf{F} & = & q\mathbf{v} \wedge \mathbf{B} = q\mathbf{v} \wedge (\nabla \wedge \mathbf{A}) & \tag{2.51} \\ & = & -q\mathbf{v} \wedge (\hat{\mathbf{z}} \wedge \nabla A) = q\left(\mathbf{v}.\hat{\mathbf{z}}\right)\nabla A \equiv qv_z \nabla A & \tag{2.52} \end{aligned}$$

Magnetic quadrupole force is identical to electric 'quadrupole' force replacing

$$\phi \leftrightarrow -Av_z \tag{2.53}$$

Consequently convergence in x-direction $\Rightarrow$ divergence in y-direction but alternating gradients give net convergence. This is basis of all "strong focussing".

2.6 Force on distributed current density

We have regarded the Lorentz force law

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) \tag{2.54}$$

as fundamental. However forces are generally measured in engineering systems via the interaction of wires or conducting bars with B-fields. Historically, of course, electricity and magnetism were based on these measurements. A current (I) is a flow of charge: Coulombs/s $\equiv$ Amp. A current density $\mathbf{j}$ is a flow of charge per unit area A/m^2. The charge is carried by particles:

$$\mathbf{j} = \sum_{species\ i} n_i \mathbf{v}_i q_i \tag{2.55}$$

Hence total force on current carriers per unit volume is

$$\mathbf{F} = \sum_i n_i q_i \left(\mathbf{v}_i \wedge \mathbf{B}\right) = \mathbf{j} \wedge \mathbf{B} \tag{2.56}$$

Also, for a fine wire carrying current I, if its area is Ω, the current density averaged across the section is

$$j = \frac{I}{\Omega} \tag{2.57}$$

Volume per unit length is Ω. And the force/unit length $= \Omega \mathbf{j} \wedge \mathbf{B} = I \times B$ perpendicular to the wire.

2.6.1 Forces on dipoles

We saw that the field of a localized current distribution, far from the currents, could be approximated as a dipole. Similarly the forces on a localized current by an external magnetic field that varies slowly in the region of current can be expressed in terms of magnetic dipole. [Same is true in electrostatics with an electric dipole].

Total force

$$\mathbf{F} = \int \mathbf{j} \wedge \mathbf{B} d^3 x' \tag{2.58}$$

where $\mathbf{B}$ is an external field that is slowly varying and so can be approximated as

$$\mathbf{B}\left(\mathbf{x}'\right) = \mathbf{B}_0 + \left(\mathbf{x}'.\nabla\right)\mathbf{B} \tag{2.59}$$

where the tensor $\nabla\mathbf{B}\ \left(\partial B_j / \partial x_i\right)$ is simply a constant (matrix). Hence

$$\begin{aligned}
\mathbf{F} &= \int \mathbf{j} \wedge \mathbf{B}_o + \mathbf{j} \wedge \left(\mathbf{x}'.\nabla\mathbf{B}\right) d^3 x' \\
&= \left(\int \mathbf{j} d^3 x'\right) \wedge \mathbf{B}_o + \int \mathbf{j} \wedge \left(\mathbf{x}'.\nabla\mathbf{B}\right) d^3 x'
\end{aligned} \tag{2.60}$$

The first term integral is zero and the second is transformed by our previous identity, which can be written for any vector $\mathbf{x}$ as

$$\mathbf{x} \wedge \int (\mathbf{x}' \wedge \mathbf{j}) \, d^3x' = 2\mathbf{x}. \int \mathbf{j}\mathbf{x}' d^3x' = -2\mathbf{x}. \int \mathbf{x}'\mathbf{j} d^3x' \tag{2.61}$$

Use the quantity $\nabla \mathbf{B}$ (constant evaluated at origin) for $\mathbf{x}$ (i.e. $x_i \leftrightarrow \frac{\partial}{\partial x_i} B_j$) giving

$$\frac{1}{2} \int (\mathbf{x}' \wedge \mathbf{j}) \, d^3x \wedge \nabla \mathbf{B} = \mathbf{m} \wedge \nabla \mathbf{B} = \int \mathbf{j} (\mathbf{x}'.\nabla) \, \mathbf{B} d^3x \tag{2.62}$$

This tensor identity is then contracted by an 'internal' cross-product $[\epsilon_{ijk} T_{jk}]$ to give the vector identity

$$(\mathbf{m} \wedge \nabla) \wedge \mathbf{B} = \int \mathbf{j} \wedge [(\mathbf{x}'.\nabla) \, \mathbf{B}] \, d^3x \tag{2.63}$$

If this internal cross product seems confusing, think of it this way. Our identity (2.61) is for *any* $\mathbf{x}$. Choose to write it for $\mathbf{x} \leftrightarrow \nabla$:

$$\mathbf{m} \wedge \nabla = \int \mathbf{j}\mathbf{x}' d^3x'.\nabla \quad . \tag{2.64}$$

Now cross multiply both sides of this equation with $\mathbf{B}$ on the right:

$$(\mathbf{m} \wedge \nabla) \wedge \mathbf{B} = \int \mathbf{j}\mathbf{x}' d^3x'.\nabla \wedge \mathbf{B} = \int \mathbf{j} \wedge [(\mathbf{x}'.\nabla) \, \mathbf{B}] \, d^3x = \mathbf{F} \tag{2.65}$$

Either way we get

$$\mathbf{F} = (\mathbf{m} \wedge \nabla) \wedge \mathbf{B} = \nabla (\mathbf{m}.\mathbf{B}) - \mathbf{m} (\nabla.\mathbf{B}) \tag{2.66}$$

(remember the ∇ operates only on $\mathbf{B}$ not $\mathbf{m}$). This is the force on a dipole:

$$\mathbf{F} = \nabla (\mathbf{m}.\mathbf{B}) \quad . \tag{2.67}$$

Total Torque (Moment of force)

is

$$\mathbf{M} = \int \mathbf{x}' \wedge (\mathbf{j} \wedge \mathbf{B}) \, d^3\mathbf{x}' \tag{2.68}$$

$$= \int \mathbf{j} (\mathbf{x}'.\mathbf{B}) - \mathbf{B} (\mathbf{x}'.\mathbf{j}) \, d^3x' \tag{2.69}$$

$\mathbf{B}$ here is (to lowest order) independent of x': $\mathbf{B}_0$ so second term is zero since

$$\int \mathbf{x}'.\mathbf{j} d^3x' = \int \frac{1}{2} \left\{ \nabla'. \left(|x'|^2 \mathbf{j} \right) - |x'|^2 \nabla'.\mathbf{j} \right\} d^3x' = 0. \tag{2.70}$$

The first term is of the standard form of our identity.

$$\mathbf{M} = \mathbf{B}. \int \mathbf{x}'\mathbf{j} d^3x' = -\frac{1}{2} \mathbf{B} \wedge \int (\mathbf{x}' \wedge \mathbf{j}) \, d^3x' \tag{2.71}$$

$$\mathbf{M} = \mathbf{m} \wedge \mathbf{B} \qquad \text{Moment on a dipole.} \tag{2.72}$$

52
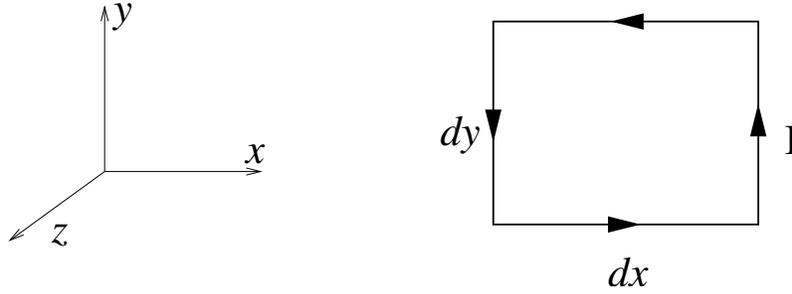
Figure 2.19: Elementary circuit for calculating magnetic force.

2.6.2 Force on an Elementary Magnetic Moment Circuit

Consider a plane rectangular circuit carrying current I having elementary area $dxdy = dA$. Regard this as a vector pointing in the z direction $\mathbf{dA}$. The force on this current in a field $\mathbf{B(r)}$ is $\mathbf{F}$ such that

$$F_x = Idy\left[B_z\left(x+dx\right) - B_z\left(x\right)\right] = Idydx\frac{\partial B_z}{\partial x} \tag{2.73}$$

$$F_y = -Idx\left[B_z\left(y+dy\right) - B_z\left(y\right)\right] = Idydx\frac{\partial B_z}{\partial y} \tag{2.74}$$

$$F_z = -Idx\left[B_y\left(y+dy\right) - B_y\left(y\right)\right] - Idy\left[B_x\left(x+dx\right) - B_x\left(x\right)\right]$$
$$= -Idxdy\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right] = Idydx\frac{\partial B_z}{\partial z} \tag{2.75}$$

(using $\nabla.\mathbf{B} = 0$). Hence, summarizing: $\mathbf{F} = Idydx\nabla B_z$. Now define $\mathbf{m} = I\mathbf{dA} = Idydx\hat{\mathbf{z}}$ and take it constant. Then clearly the force can be written

$$\mathbf{F} = \nabla\left(\mathbf{B.m}\right) \tag{2.76}$$

or strictly $(\nabla\mathbf{B}).\mathbf{m}$.

2.6.3 Example

Small bar magnet: archetype of dipole. In uniform $\mathbf{B}$ feels just a torque aligning it with $\mathbf{B}$.
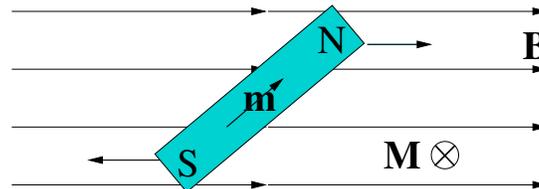
Figure 2.20: Moment on a bar magnet in a uniform field.

In a uniform field, no net force.

Non-uniform field: If magnet takes its natural resting direction, **m** parallel to **B**, force is
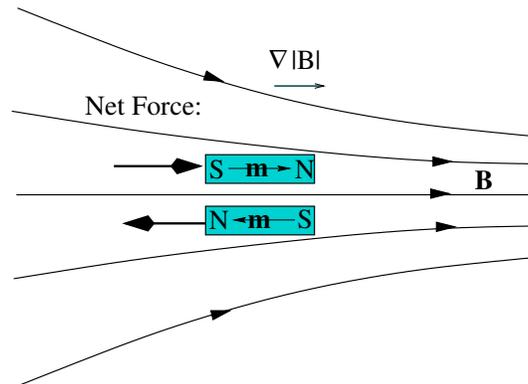
Figure 2.21: A magnetic moment in the form of a bar magnet is attracted or repelled toward the stronger field region, depending on its orientation.

$$\mathbf{F} = m\nabla|B| \qquad (2.77)$$

A bar magnet is attracted to high field. Alternatively if **m** parallel to *minus* **B** the magnet points other way

$$\mathbf{F} = -m\nabla|B| \qquad \text{repelled from high } |\mathbf{B}|. \qquad (2.78)$$

Same would be true for an elementary circuit dipole. It is attracted/repelled according to
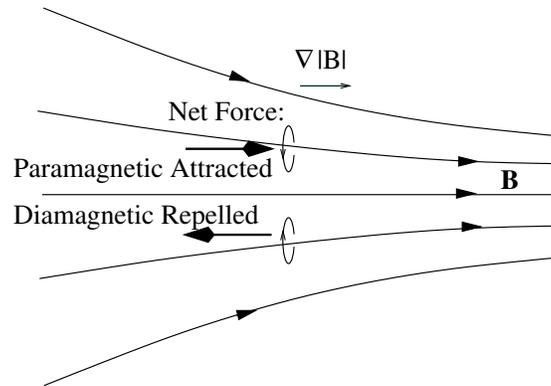
Figure 2.22: Elementary circuit acting as a dipole experiences a force in a non-uniform magnetic field.

whether it acts to increase or decrease B locally. A charged particle moving in its Larmor orbit is always <u>diamagnetic</u>: repelled from high $|B|$.

2.6.4 Intuition

There is something slightly non-intuitive about the "natural" behavior of an elementary wire circuit and a particle orbit considered as similar to this elementary circuit. Their currents
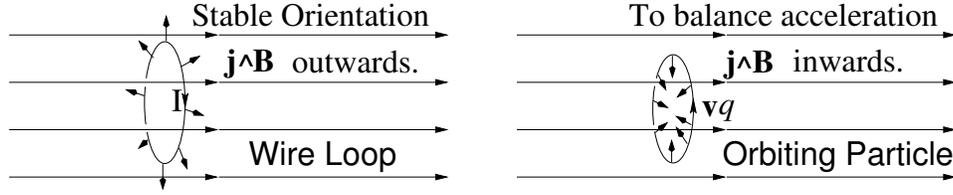
Figure 2.23: Difference between a wire loop and a particle orbit in their "natural" orientation.

flow in opposite directions when the wire is in its stable orientation. The reason is that the strength of the wire sustains it against the outward magnetic expansion force, while the particle needs an inward force to cause the centripetal acceleration.

2.6.5 Angular Momentum

If the local current is made up of particles having a constant ratio of charge to mass: q/M say (Notational accident $\mathbf{m}$ is magnetic moment). Then the angular momentum is $\mathbf{L} = \sum_i M_i \mathbf{x}_i \wedge \mathbf{v}$ and magnetic moment is $\mathbf{m} = \frac{1}{2} \sum q_i \mathbf{x}_i \wedge \mathbf{v}_i$. So

$$\mathbf{m} = \frac{q}{2M}\mathbf{L}. \qquad \text{``Classical''} \tag{2.79}$$

This would also be true for a continuous body with constant (charge density)/(mass density) (ρ/ρ_m). Elementary particles, e.g. electrons etc., have 'spin' with moments $\mathbf{m}, \mathbf{L}$. However they do not obey the above equation. Instead

$$\mathbf{m} = g\frac{q}{2M}\mathbf{L} \tag{2.80}$$

with the Landé g-factor ($\simeq 2$ for electrons). This is attributed to quantum and relativistic effects. However the "classical" value might not occur if ρ/ρ_m were not constant. So we should not be surprised that g is not exactly 1 for particles' spin.

2.6.6 Precession of a Magnetic Dipole (formed from charged particle)

The result of a torque $\mathbf{m} \wedge \mathbf{B}$ is a change in angular momentum. Since $\mathbf{m} = g\mathbf{L}q/2M$ we have

$$\frac{d\mathbf{L}}{dt} = \mathbf{m} \wedge \mathbf{B} = g\frac{q}{2M}\left(\mathbf{L} \wedge \mathbf{B}\right)) \tag{2.81}$$

This is the equation of a circle around B. [Compare with orbit equation $\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \wedge \mathbf{B}$]. The direction of $\mathbf{L}$ precesses like a tilted 'top' around direction of B with a frequency

$$\omega = g\frac{qB}{2M} \tag{2.82}$$
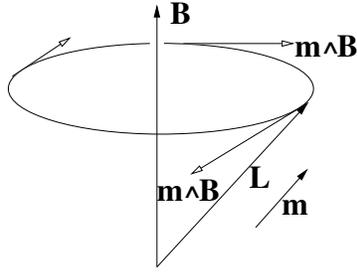
Figure 2.24: Precession of an angular momentum L and aligned magnetic moment $\mathbf{m}$ about the magnetic field.

For an electron $(g = 2)$ this is equal to the cyclotron frequency. For protons $g = 2 \times 2.79$ [Written like this because spin is $\frac{1}{2}$]. For neutrons $g = 2 \times (-1.93)$.
Precession frequency is thus

$$f = \frac{\omega_{electron}}{2\pi} \quad = \quad (28\text{GHz}) \times (B/\text{Tesla}) \tag{2.83}$$

$$\frac{\omega_{proton}}{2\pi} \quad = \quad (43\text{MHz}) \times (B/\text{Tesla}) \tag{2.84}$$

This is the (classical) basis of Nuclear Magnetic Resonance but of course that really needs QM.

Chapter 3

Dynamics of the Electromagnetic Fields

3.1 Maxwell Displacement Current

In the early 1860s (during the American civil war!) electricity including induction was well established experimentally. A big row was going on about theory. The warring camps were divided into the

- Action-at-a-distance advocates and the

- Field-theory advocates.

James Clerk Maxwell was firmly in the field-theory camp. He invented mechanical analogies for the behavior of the fields locally in space and how the electric and magnetic influences were carried through space by invisible circulating cogs. Being a consumate mathematician he also formulated differential equations to describe the fields. In modern notation, they would (in 1860) have read:

$$
\begin{aligned}
\nabla.\mathbf{E} &= \frac{\rho}{\epsilon_0} && \text{Coulomb's Law} \\
\nabla \wedge \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{Faraday's Law} \\
\nabla.\mathbf{B} &= 0 \\
\nabla \wedge \mathbf{B} &= \mu_0 \mathbf{j} && \text{Ampere's Law. (Quasi-static)}
\end{aligned}
\tag{3.1}
$$

Maxwell's stroke of genius was to realize that this set of equations is <u>inconsistent</u> with charge conservation. In particular it is the quasi-static form of Ampere's law that has a problem. Taking its divergence

$$
\mu_0 \nabla.\mathbf{j} = \nabla.\left(\nabla \wedge \mathbf{B}\right) = 0
\tag{3.2}
$$

(because divergence of a curl is <u>zero</u>). This is fine for a static situation, but can't work for a time-varying one. Conservation of charge in time-dependent case is

$$
\nabla.\mathbf{j} = -\frac{\partial \rho}{\partial t} \qquad \underline{\text{not zero}}.
\tag{3.3}
$$

The problem can be fixed by adding an extra term to Ampere's law because

$$\nabla.\mathbf{j} + \frac{\partial \rho}{\partial t} = \nabla.\mathbf{j} + \frac{\partial}{\partial t}\epsilon_0\nabla.\mathbf{E} = \nabla.\left(\mathbf{j} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \qquad (3.4)$$

Therefore Ampere's law is consistent with charge conservation only if it is really to be written with the quantity $(\mathbf{j} + \epsilon_0 \partial\mathbf{E}/\partial t)$ replacing $\mathbf{j}$. i.e.

$$\nabla \wedge \mathbf{B} = \mu_0\left(\mathbf{j} + \epsilon_0\frac{\partial E}{\partial t}\right) = \mu_0\mathbf{j} + \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} \qquad (3.5)$$

The term $\epsilon_0\partial\mathbf{E}/\partial t$ is called Maxwell's Displacement Current. There was, at the time, no experimental evidence for this term; still, the equations demanded it, as Maxwell saw. The addition makes the source-free forms (where ρ and $\mathbf{j}$ are zero) beautifully symmetric

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad ; \qquad \nabla \wedge \mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \qquad . \qquad (3.6)$$

Meaning a changing $\mathbf{B}$-field induces an $\mathbf{E}$-field (induction) but <u>also</u> a changing $\mathbf{E}$-field induces a $\mathbf{B}$-field. The extra term is also responsible, as we shall shortly see, for electromagnetic waves. Despite their elegance, the Maxwell equations did not settle the argument. The equations can be rewritten in integral form, similar to what the action-at-a-distance advocates wanted. However the action, it now became clear, could not be regarded as instantaneous. Moreover EM waves were not detected unequivocally for another 23 years!

3.2 Field Dynamics, Energy and Momentum

3.2.1 Introduction

Suppose we take a capacitor and charge it up using a power supply.   During charging, a
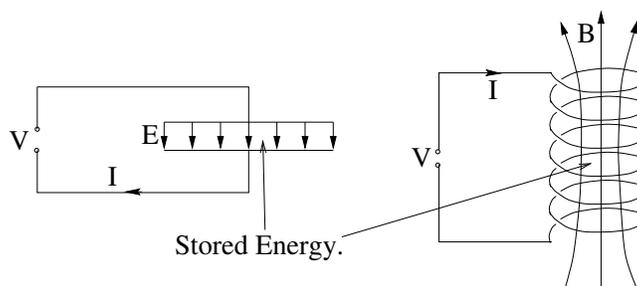
Figure 3.1: Energy obtained from the power supply in "charging up" a capacitor or inductor is stored in the electromagnetic field.

current I is flowing and at any instant potential is $V = \frac{Q}{C}$

$$Q = \int I\,dt \qquad , \qquad I = \frac{dQ}{dt} \qquad . \qquad (3.7)$$

Thus the power supply does work at rate IV (per sec) and total work done is

$$\int IV dt \;=\; \int V \frac{dQ}{dt} dt = \int V dQ \tag{3.8}$$

$$=\; \int \frac{Q}{C} dQ = \frac{Q^2}{2C} \qquad \text{Starting at Q = 0).} \tag{3.9}$$

Where has this energy gone? Answer: into the electric field. The E-field within the capacitor stores energy with a volumetric energy density we shall calculate Now consider 'charging' an inductor with self-inductance L.

$$V = L\frac{dI}{dt} \quad . \quad \text{Work done} \qquad \int V I dt = \int L I dI = \frac{LI^2}{2} \tag{3.10}$$

Where has this energy gone? The magnetic field.

3.2.2 Poynting's Theorem: Energy Conservation

How do we know, or show, that EM fields store and transport energy? Formally from a theorem derived from Maxwell's Equations. Energy dissipation is the rate of doing work by fields on particles. Energy is transferred from fields to particles (and then often turned by some randomizing process into 'heat'). Magnetic field does <u>no</u> work on particles because $\mathbf{F} \perp \mathbf{v}$. Electric field rate of doing work on a single particle is

$$\mathbf{F}.\mathbf{v} = q\mathbf{E}.\mathbf{v} \tag{3.11}$$

Average rate of doing work on all particles in a volume V is

$$\sum_{j \ in \ V} q_j \mathbf{E}\left(\mathbf{x}_j\right).\mathbf{v}_j \tag{3.12}$$

so for an elemental volume dV such that $\mathbf{E}$ can be taken uniform across the volume, rate of work is

$$\mathbf{E}.\sum_{j} q_j \mathbf{v}_j = \mathbf{E}.\langle qn\mathbf{v}\rangle dV \tag{3.13}$$

where n is the number/unit volume = density. The average $\langle qn\mathbf{v}\rangle$ is just the current-density, $\mathbf{j}$. Hence the energy dissipation (= work done on particles) rate (i.e. power) per unit volume is

$$\mathbf{E}.\mathbf{j} \tag{3.14}$$

[Of course this is a localized form of the circuit equation $P = VI$]. We have the energy dissipation rate density $\mathbf{E}.\mathbf{j}$, but now we can express this in terms of the fields by using Ampere's law:

$$\mathbf{E}.\mathbf{j} = \mathbf{E}.\frac{1}{\mu_0}\left[\nabla \wedge \mathbf{B} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right] \tag{3.15}$$

Now we convert the form of the first term, $\mathbf{E}.(\nabla \wedge \mathbf{B})$ using a vector identity

$$\nabla.(\mathbf{E} \wedge \mathbf{B}) = \mathbf{B}.(\nabla \wedge \mathbf{E}) - \mathbf{E}.(\nabla \wedge \mathbf{B}) \tag{3.16}$$

so

$$\mathbf{E.j} = \frac{1}{\mu_0}\left\{-\nabla.(\mathbf{E} \wedge \mathbf{B}) + \mathbf{B}.(\nabla \wedge \mathbf{E}) - \mu_0\epsilon_0\mathbf{E}.\frac{\partial \mathbf{E}}{\partial t}\right\} \tag{3.17}$$

Then, using Faraday's law $\nabla \wedge \mathbf{E} = -\partial \mathbf{B}/\partial t$ we have

$$\mathbf{E.j} = -\frac{1}{\mu_0}\left\{\nabla.(\mathbf{E} \wedge \mathbf{B}) + \mathbf{B}.\frac{\partial \mathbf{B}}{\partial t} + \mu_0\epsilon_0\mathbf{E}.\frac{\partial \mathbf{E}}{\partial t}\right\} \tag{3.18}$$

Note that if we had used the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ (which in vacuo are $\mathbf{D} = \epsilon_0\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu_0$) we would have obtained

$$\mathbf{E.j} = -\left\{\nabla.(\mathbf{E} \wedge \mathbf{H}) + \mathbf{H}.\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t}\right\} \tag{3.19}$$

which is entirely equivalent in the vacuum but subtly different for dielectric and magnetic media in the way energy accounting is done. Notice that this can be written (for vacuum)

$$\mathbf{E.j} = -\left\{\nabla.(\mathbf{E} \wedge \mathbf{B}/\mu_0) + \frac{\partial}{\partial t}\tfrac{1}{2}\left(\mathbf{B.B}/\mu_0 + \epsilon_0\mathbf{E.E}\right)\right\} \tag{3.20}$$

Although it may not be obvious, this is now in the form of an energy conservation law. The physical meaning can be made obvious by considering an arbitrary volume V, with surface A.

$$\int_V \mathbf{E.j}d^3x = -\int_V \nabla.(\mathbf{E} \wedge \mathbf{B}/\mu_0) + \frac{\partial}{\partial t}\frac{1}{2}\left[\mathbf{B.B}/\mu_0 + \epsilon_0\mathbf{E.E}\right]d^3x \tag{3.21}$$

$$= -\int_A (\mathbf{E} \wedge \mathbf{B}/\mu_0).\mathbf{dA} - \frac{\partial}{\partial t}\int_V \frac{1}{2}\left[B^2/\mu_0 + \epsilon_0 E^2\right]d^3x \tag{3.22}$$

This says that the total rate of work on particles in V is equal to minus the rate of change
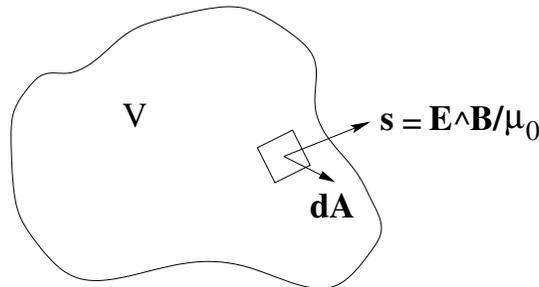
Figure 3.2: Integral of Poynting flux over surface of V.

of the integral over V of

$$\frac{1}{2}\left[B^2/\mu_0 + \epsilon_0 E^2\right] \tag{3.23}$$

minus the <u>flux</u> of the vector $\mathbf{E}\wedge\mathbf{B}/\mu_0$ across the surface A. Physically this says that $\frac{1}{2}\left[B^2/\mu_0 + \epsilon_0 E^2\right]$ must be considered to be the <u>electromagnetic energy density</u> in the volume V and the quantity $\mathbf{E}\wedge\mathbf{B}/\mu_0$ must be considered to be the <u>energy flux density</u> (across any surface). $\mathbf{s} = \mathbf{E}\wedge\mathbf{B}/\mu_0 = \mathbf{E}\wedge\mathbf{H}$ is called the "Poynting Vector". If we write $w \equiv \frac{1}{2}\left[B^2/\mu_0 + \epsilon_0 E^2\right]$ the energy density, then Poynting's theorem is:

$$\mathbf{E.j} = -\nabla.\mathbf{s} - \frac{\partial w}{\partial t}. \tag{3.24}$$

The significance of the indentification of field energy density and energy flux density is immense. Even today we tend to think of electric power as being carried some how "in" the conducting cables. But if we understand Poynting's theorem and EM theory we realize the power is carried <u>by the fields</u> as represented by $\mathbf{E}\wedge\mathbf{B}/\mu_0$. Not by the electrons in the conductor, although they do carry the current.

3.2.3 Momentum Conservation

The rate at which fields transfer momentum to particles is equal to the EM force $q(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})$ and the force density is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j}\wedge\mathbf{B} \quad . \tag{3.25}$$

Use Maxwell's equations to eliminate ρ and $\mathbf{j}$ in favor of fields:

$$\rho = \epsilon_0\nabla.\mathbf{E} \quad ; \quad \mathbf{j} = \frac{1}{\mu_0}\left(\nabla\wedge\mathbf{B} - \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad . \tag{3.26}$$

Then

$$
\begin{aligned}
\mathbf{f} &= \epsilon_0\left[\mathbf{E}\left(\nabla.\mathbf{E}\right) + \mathbf{B}\wedge\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{\epsilon_0\mu_0}\mathbf{B}\wedge\left(\nabla\wedge\mathbf{B}\right)\right] \\
&= -\frac{\partial}{\partial t}\left(\epsilon_0\mathbf{E}\wedge\mathbf{B}\right) + \epsilon_0\left[-\frac{\partial\mathbf{B}}{\partial t}\wedge\mathbf{E} + \mathbf{E}\left(\nabla.\mathbf{E}\right) - \frac{1}{\epsilon_0\mu_0}\left\{\frac{1}{2}\nabla.\left(\mathbf{B.B}\right) - \left(\mathbf{B.\nabla}\right)\mathbf{B}\right\}\right] \\
&= -\frac{\partial}{\partial t}\left(\epsilon_0\mathbf{E}\wedge\mathbf{B}\right) + \epsilon_0\left[\left(\nabla\wedge\mathbf{E}\right)\wedge\mathbf{E} + \mathbf{E}\left(\nabla.\mathbf{E}\right)\right] - \frac{1}{\mu_0}\left[\frac{1}{2}\nabla B^2 - \left(\mathbf{B.\nabla}\right)\mathbf{B}\right] \\
&= -\frac{\partial}{\partial t}\left(\epsilon_0\mathbf{E}\wedge\mathbf{B}\right) + \epsilon_0\left[\left(\mathbf{E.\nabla}\right)\mathbf{E} - \frac{1}{2}\nabla\left(\mathbf{E.E}\right) + \mathbf{E}\left(\nabla.\mathbf{E}\right)\right] \\
&\quad + \frac{1}{\mu_0}\left[\left(\mathbf{B.\nabla}\right)\mathbf{B} - \frac{1}{2}\nabla\left(\mathbf{B.B}\right) + \mathbf{B}\left(\nabla.\mathbf{B}\right)\right]
\end{aligned}
\tag{3.27}
$$

Now the last two terms can be written as the divergence $\nabla.\mathbf{T}$ of a tensor quantity

$$\mathbf{T} \equiv \epsilon_0\left[\mathbf{EE} - \frac{1}{2}E^2\mathbf{I}\right] + \frac{1}{\mu_0}\left[\mathbf{BB} - \frac{1}{2}B^2\mathbf{I}\right] \tag{3.28}$$

where $\mathbf{I}$ denotes the unit tensor. In suffix notation

$$T_{ij} = \epsilon_0 \left[E_i E_j - \frac{1}{2} E^2 \delta_{ij} \right] + \frac{1}{\mu_0} \left[B_i B_j - \frac{1}{2} B \delta_{ij} \right] \quad . \tag{3.29}$$

$\mathbf{T}$ is called the "Maxwell Stress Tensor." And the force equation (Momentum conservation) becomes

$$\mathbf{f} = -\frac{\partial}{\partial t} \left(\epsilon_0 \mathbf{E} \wedge \mathbf{B} \right) + \nabla.\mathbf{T} \tag{3.30}$$

which is, like Poynting's theorem, in conservation form. Just as before, the physical meaning is seen by integration over a volume, finding that

$$\epsilon_0 \mathbf{E} \wedge \mathbf{B} = \frac{1}{c^2} \mathbf{E} \wedge \mathbf{B}/\mu_0 = \frac{1}{c^2} \mathbf{s} \tag{3.31}$$

is the volumetric field momentum density, and $\mathbf{T}$ is the force per unit area at a surface, i.e. the <u>stress</u>. Electromagnetic fields thus carry momentum density that is $\frac{1}{c^2}$ times their energy flux density, where we have used the fact that $\epsilon_0 \mu_0 = \frac{1}{c^2}$.

By concentrating on $\mathbf{E}$ and $\mathbf{B}$, leaving all charges and currents explicit in ρ and $\mathbf{j}$ we exclude all "mechanical" energy and momentum associated e.g., with motion of or polarization of atoms or molecules or their constituent parts. (Even though that energy or momentum might be electromagnetic if we were dealing with $\mathbf{E}$ and $\mathbf{B}$ averaged across all atoms). Most confusion with energy and momentum in EM problems arises from not being clear about what counts in the <u>EM</u> energy/momentum versus particle energy/momentum.

3.3 Inductance, Energy, and Magnet Stresses

Poynting's theorem formalizes the observation we already made that the energy required to 'charge up' an inductance (i.e. raise the current in it) is stored in the magnetic field. We now know that the energy density is $B^2/2\mu_0$ (in vacuuo) or $\frac{1}{2}\mathbf{B}.\mathbf{H}$ in a linear magnetic medium (but most magnetic materials are <u>not</u> linear). Similarly the energy density stored in the electric field of a capacitor is $\epsilon_0 E^2/2$, or $\frac{1}{2}\mathbf{E}\mathbf{D}$. We also found that the force density associated with B was governed by a tensor $\frac{1}{\mu_0} \left[\mathbf{B}\mathbf{B} - \frac{1}{2} B^2 \mathbf{I} \right]$ the second term of which is of the same form as the energy density $B^2/2\mu_0$. There is a fundamental reason for this relationship that we can show by thinking about forces on magnets.

3.3.1 Relation between energy density and magnetic pressure in a solenoid

Consider a solenoid formed by a current flowing azimuthally. It is easy to show that the EM force $\mathbf{j} \wedge \mathbf{B}$ is always outward. Actually we can't just take the total current and multiply by internal field to get the force, because $\mathbf{j}$ and B vary through the magnet. $B = 0$ outside. Instead of doing the integral of jB, let's calculate the force by the method of "virtual work". This involves imagining a small incremental motion, calculating energy changes and putting
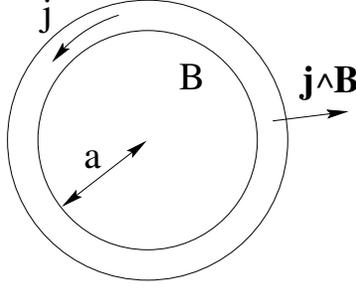
Figure 3.3: Force on a solenoid magnet.

them equal to the work done $F.dx$. So ignore the thickness of conductor, and consider an expansion of the initial radius a by a small increment da. The stored magnetic energy (per unit axial length) $\int_o^a \frac{B^2}{2\mu_0} 2\pi r dr$ changes because a changes, and because B (possibly) changes. Actually, whether or not B changes depends on the external circuit attached to the solenoid through which the current flows. Let us choose to suppose that that circuit acts to keep the current constant so that B is constant. We need to calculate how much energy the circuit provides to the inductance. This requires the voltage during the expansion. Remembering Faraday's law,

$$V = \oint \mathbf{E}.\mathbf{dl} = -\frac{\partial \Phi}{\partial t} \tag{3.32}$$

If B is constant, $\Phi = \pi a^2 B$ so

$$\frac{d\Phi}{dt} = B 2\pi a \frac{da}{dt} \tag{3.33}$$

Hence the voltage induced in a single turn is

$$V = -B 2\pi a \frac{da}{dt} \quad . \tag{3.34}$$

The current per unit length needed to give B is $\mu_0 J = B$. So the work done by the external circuit is (per unit length)

$$\int -VJdt \quad = \quad \int \frac{B}{\mu_0} B 2\pi a \frac{da}{dt} dt \tag{3.35}$$

$$dW_{circuit} \quad = \quad \frac{B^2}{\mu_0} 2\pi a da \tag{3.36}$$

(for a small incremental da). The change in stored magnetic energy is

$$\frac{B^2}{2\mu_0} d^3 x = \frac{B^2}{2\mu_0} 2\pi a da = dW_{magnetic} \quad . \tag{3.37}$$

Then energy conservation is that

$$dW_{circuit} = dW_{magnetic} + dW_{mechanical} \tag{3.38}$$

where $dW_{mechanical} = 2\pi ada.P$ and P is the force per unit length axially, per unit length azimuthally, i.e. the force per unit area or <u>pressure</u>. Substituting

$$\frac{B^2}{\mu_0}2\pi ada = \frac{B^2}{2\mu_0}2\pi ada + P\,2\pi ada \tag{3.39}$$

Hence

$$P = \frac{B^2}{2\mu_0} \tag{3.40}$$

is the outward pressure exerted by the B-field on the magnet. Notice that we never invoked any force law such as the Maxwell stress tensor, we only used our knowledge of magnetic energy density. Also, the final result does not depend on our assumption about the external circuit. We could have assumed anything we liked. If we did the energy counting correctly, we would get the same force result. Also, if we assume the B-field distribution in the conductor does not change, then we don't need to know what it is to obtain this result. So the total force/area on the magnet doesn't depend on the current/field distribution in the magnet, provided the magnet is thin so that any energy stored in the magnetic field *in the thickness of the magnet* is small. Magnetic pressure is large for high fields.

$$P = \frac{B^2}{2\mu_0} = 4.0 \times 10^5 B^2 \text{ Pa} \qquad = 4B^2 \text{ bar.} \tag{3.41}$$

1T magnetic field pressure is 4 bar ($\sim$ atmospheres). 10T B-field pressure is 40MPa (c.f. yield strength of hard copper $\sim 300MPa$). For a thin cylinder, the stress (hoop stress) induced by a pressure P is

$$\frac{a}{t}P \tag{3.42}$$

a = radius, t = thickness. High field magnets have to be 'chunky' and even then soon run
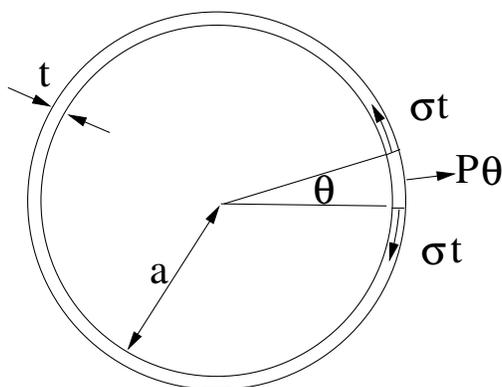
Figure 3.4: The hoop stress, σ in a thin cylinder balances the outward pressure, P.

into stress limits.

3.4 Potentials for Time Varying Fields

Electrostatic and Magnetostatic problems are most easily solved using the potentials ϕ and $\mathbf{A}$. These potentials are also critical in time varying situations and general equations can be found as follows for the complete Maxwell equations.

$$\nabla.\mathbf{B} = 0 \qquad \text{means that} \qquad \mathbf{B} = \nabla \wedge \mathbf{A} \tag{3.43}$$

is still a valid representation. Then

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla \wedge \mathbf{A} = -\nabla \wedge \frac{\partial \mathbf{A}}{\partial t} \tag{3.44}$$

So

$$\nabla \wedge \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \quad . \tag{3.45}$$

Therefore

$$\mathbf{E} + \dot{\mathbf{A}} \tag{3.46}$$

can be written as the gradient

$$\mathbf{E} + \dot{\mathbf{A}} = -\nabla\phi \qquad \text{or} \qquad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad . \tag{3.47}$$

Then Coulomb's law becomes

$$\frac{\rho}{\epsilon_0} = \nabla.\mathbf{E} = -\nabla.\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right) = -\nabla^2\phi - \frac{\partial}{\partial t}\left(\nabla.\mathbf{A}\right) \tag{3.48}$$

and Ampere's Law

$$\begin{aligned} \mu_0\mathbf{j} &= \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\frac{1}{c^2} = \nabla \wedge (\nabla \wedge \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial \nabla\phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \\ &= \nabla\left(\nabla.\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) - \nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \end{aligned} \tag{3.49}$$

Now remember that there is an arbitrariness to our choice of $\mathbf{A}$ since only its curl is equal to $\mathbf{B}$. In point of fact we can choose $\nabla.\mathbf{A}$ to be whatever we want. One choice, Coulomb gauge, was $\nabla.\mathbf{A} = 0$. "Lorentz Gauge":

$$\nabla.\mathbf{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t} \quad . \tag{3.50}$$

Then

$$\nabla^2\phi \quad - \quad \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{3.51}$$

$$\nabla^2\mathbf{A} \quad - \quad \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} \tag{3.52}$$

Maxwell's equations are completely equivalent to these wave equations with sources. (Plus Lorentz gauge condition.) Considered in this gauge, we see that the EM influence of ρ or $\mathbf{j}$ does not act instantaneously at-a-distance. Instead the influence has to propagate from the sources at the speed of light, c.

3.4.1   General Solutions

We want to find the general solution to these equations. We work just on the ϕ equation because it is scalar. Its solution will generalize immediately. First let's discuss the homogeneous equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{3.53}$$

which is satisfied wherever $\rho = 0$ (in vacuo). Also remember we can add any solution of this equation to a solution of the <u>in</u>homogeneous eq. One solution type is plane waves, i.e. things that vary in only one direction. If we choose axes such that $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$ then equation is 1-d:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{3.54}$$

The general solutions of this equation are

$$\phi(x,t) = f(x \pm ct) \quad . \tag{3.55}$$

That is, arbitrary shaped functions that move toward either increasing or decreasing x, preserving their shape. For our problem the more interesting case is spherically symmetric
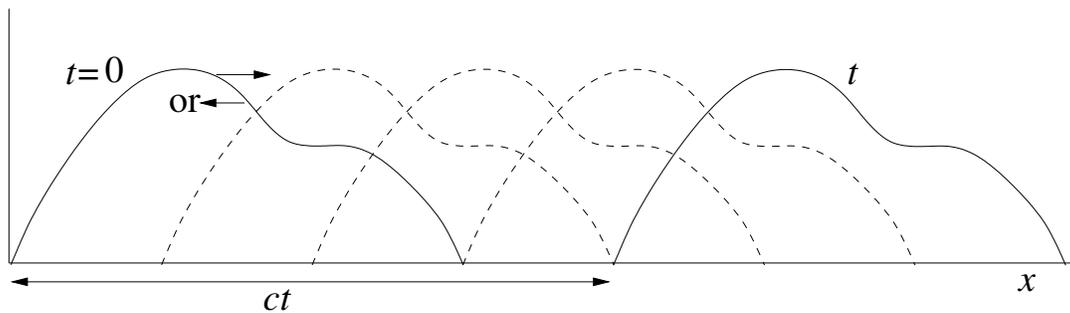
Figure 3.5: Arbitrary solution of the one-dimensional wave equation.

waves. That is, in spherical coordinates (r, θ, χ) solutions such that $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\chi} = 0$. Then

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{3.56}$$

Make the substitution

$$\phi = \frac{u}{r} \tag{3.57}$$

then

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial \phi}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{\partial u}{\partial r}\frac{1}{r} - \frac{u}{r^2}\right) = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} - u\right)\right]$$

$$= \frac{1}{r^2}\left[\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} - \frac{\partial u}{\partial r}\right] = \frac{1}{r}\frac{\partial^2 u}{\partial r^2} \tag{3.58}$$

so

$$\frac{\partial^2 u}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{3.59}$$

So u satisfies the 1-d (planar) wave equation, with general solution $f(r \pm ct)$. Hence

$$\phi = \frac{f(r \pm ct)}{r} \tag{3.60}$$

is the general solution of the homogeneous wave equation that is spherically symmetric. Expanding (- sign) or converging (+ sign) radial waves. Actually this spherically symmetric solution <u>doesn't</u> satisfy the homogeneous equation at $r = 0$ because of the singularity there. And in fact we already know from the static problem that

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r}) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{3.61}$$

(taking the center of coordinate space to be $\mathbf{x}'$). Therefore, if f is everywhere non-singular, then

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{f(r \pm ct)}{r} = -4\pi f(\pm ct)\delta(\mathbf{x} - \mathbf{x}') \tag{3.62}$$

So our solution $\phi = \frac{f(r \pm ct)}{r}$ is really the solution of the (mathematical) problem of calculating the potential of a *time-varying point charge* at position x', that is, of a charge density $\rho = q(t)\delta(\mathbf{x} - \mathbf{x}')$, where the charge is related to f by

$$f(\pm ct) = \frac{q(t)}{4\pi\epsilon_0} \qquad \text{and hence} \qquad f(r \pm ct) = \frac{q(t \pm r/c)}{4\pi\epsilon_0} \quad . \tag{3.63}$$

In short, the potential at position $\mathbf{x}$ due to a time-varying charge of magnitude $q(t)$ at $\mathbf{x}'$ is

$$\phi(\mathbf{x}, t) = \frac{q(t \pm |\mathbf{x} - \mathbf{x}'|/c)}{4\pi\epsilon_0|\mathbf{x} - \mathbf{x}'|} \tag{3.64}$$

By considering the case where the charge is a delta-function in time, as well as space, $q(t) = \delta(t - t')$, we see that the Green Function $G(\mathbf{x}, t|\mathbf{x}', t')$ (in time and space) for the operator

$$\mathcal{L} \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad , \tag{3.65}$$

namely the function that solves $\mathcal{L}G(\mathbf{x}, t|x', t') = \delta(\mathbf{x} - \mathbf{x}') \times \delta(t - t')$, is

$$G(\mathbf{x}, t|\mathbf{x}', t') = \frac{-\delta\left(t \pm \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t'\right)}{4\pi|\mathbf{x} - \mathbf{x}'|} \tag{3.66}$$

and that the general solution of the electrostatic potential equation,

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{3.67}$$

is therefore

$$\phi\left(\mathbf{x}, t\right) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\mathbf{x}', t \pm \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad . \tag{3.68}$$

3.5 Advanced and Retarded Solutions

Notice we still have this $\pm$ sign in our potential solution. If we take the $-$ sign, then the integrand is

$$\frac{\rho\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} \quad . \tag{3.69}$$

This says that the contribution to our potential at $\mathbf{x}, t$ from a charge density at $\mathbf{x}'$ depends only on the value of that charge at the time

$$t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \tag{3.70}$$

This is earlier by the time it takes the EM influence to propagate from $\mathbf{x}'$ to $\mathbf{x}$, i.e. by $\frac{|\mathbf{x}-\mathbf{x}'|}{c}$. A potential based on this sign is called the "retarded" potential because the influence arrives later than the charge: retarded. The time t' is called "retarded time" very often, despite being earlier. [In English retarded $\equiv$ delayed.] If we were to take the $+$ sign we would have the very peculiar result that the influence (i.e. potential) would depend on the charge density at a _later_ time. The "advanced" solution from

$$\frac{\rho\left(\mathbf{x}', t + \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} \tag{3.71}$$

thus violates our ideas of causality. We generally hold that an effect (potential) can only arise from a cause (charge) if the cause is earlier in time. For that reason, the advanced potential is discarded as "unphysical" but the justification for this choice is mysterious, bound up in philosophical discussions of the arrow of time.

Having obtained the general solution for the scalar wave equation with sources, we can immediately apply it to each vector component of the equation for $\mathbf{A}$; so in summary:

$$\phi\left(\mathbf{x}, t\right) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\mathbf{x}', t'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x \tag{3.72}$$

$$\mathbf{A}\left(\mathbf{x}, t\right) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}\left(\mathbf{x}', t'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{3.73}$$

with $t' = t - |\mathbf{x}-\mathbf{x}'|/c$.

Often the notation $[\![f]\!]$ is used to denote evaluation of any <u>function</u> f at <u>retarded</u> time. You must be extremely careful with taking differentials of retarded quantities because there is dependence on $\mathbf{x}$ not only in the space but also in the time argument. Thus, for example

$$\nabla' f\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) = \nabla'[\![f]\!] \neq [\![\nabla' f]\!] \tag{3.74}$$

the retarded value of a gradient is not equal to the gradient of the retarded value.

In general

$$\nabla'\left\{f\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)\right\} = [\![\nabla' f]\!] + \nabla'\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)\left[\!\left[\frac{\partial f}{\partial t}\right]\!\right] \tag{3.75}$$

$$= [\![\nabla' f]\!] + \frac{\mathbf{x} - \mathbf{x}'}{c|\mathbf{x} - \mathbf{x}'|}\left[\!\left[\frac{\partial f}{\partial t}\right]\!\right] = \nabla'[\![f]\!] \tag{3.76}$$

$$\text{and} \quad \nabla' \wedge [\![\mathbf{v}]\!] = [\![\nabla' \wedge \mathbf{v}]\!] + \frac{\mathbf{x} - \mathbf{x}'}{c|\mathbf{x} - \mathbf{x}'|} \wedge \left[\!\left[\frac{\partial \mathbf{v}}{\partial t}\right]\!\right] \quad . \tag{3.77}$$

These can be approached in an equivalent way by asking what is $\nabla' t'$? First, what exactly does this denote? It means take the derivative with respect to $\mathbf{x}'$, keeping $\mathbf{x}$ and t constant.

$$\nabla' t' = \nabla'\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) = \frac{\mathbf{x} - \mathbf{x}'}{c|\mathbf{x} - \mathbf{x}'|}. \tag{3.78}$$

Then, f is a function of its two arguments $f(\mathbf{a}_1, a_2)$, both of which may depend upon $\mathbf{x}'$, giving

$$\nabla' f = \nabla' \mathbf{a}_1 \frac{\partial f}{\partial \mathbf{a}_1} + \nabla' a_2 \frac{\partial f}{\partial a_2}. \tag{3.79}$$

When those arguments are $\mathbf{a}_1 = \mathbf{x}'$, $a_2 = t'$, the above expressions result.

Chapter 4

Radiation By Moving Charges

4.1 Potentials and Fields of a moving point charge

The general solution

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\![\rho]\!]}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \tag{4.1}$$

<u>Looks</u> as if it will give the result for a point charge directly in the same way as the static solution. For a stationary point charge $\rho(\mathbf{x}') = q\delta(\mathbf{x}' - \mathbf{r})$, where $\mathbf{r}$ is the charge position, $\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x}-\mathbf{r}|}$. For brevity let's write $\mathbf{R} \equiv \mathbf{x} - \mathbf{r}$. One <u>might think</u> for a moving charge $\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}[\![1/R]\!]$ but this is <u>incorrect</u>. We haven't taken care with derivatives etc. of
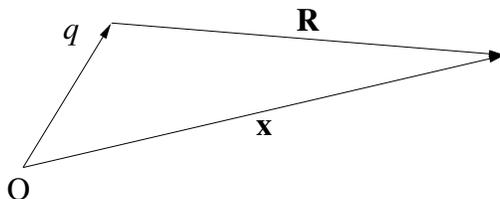
Figure 4.1: Vector coordinates of charge and field point.

retarded quantities. Let's go carefully! The charge density is $\rho(x, t) = q\delta(\mathbf{x} - \mathbf{r}(t))$ where $\mathbf{r}$ is now allowed to vary with time so we want

$$
\begin{aligned}
\phi(\mathbf{x}, t) &= \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{x}' - \mathbf{r}(t'))}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \quad, \quad t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{r}(t')|} \int \delta(\mathbf{x}' - \mathbf{r}(t')) d^3 x' \tag{4.2}
\end{aligned}
$$

where we make use of the fact that the delta function is non-zero only where its argument is zero, so all the contribution to the integral comes from the place where $\mathbf{x}' = \mathbf{r}(t')$, which is where the particle is at retarded time i.e.

$$\mathbf{r}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) \tag{4.3}$$

[This requires self-referential notation which is one reason we write it $[\![\mathbf{r}]\!]$.] Now we have to do the integral $\int \delta\left(\mathbf{x}' - \mathbf{r}\left(t'\right)\right) d^3 x'$. This is <u>not</u> unity because $\mathbf{x}'$ appears inside $\mathbf{r}(t')$ as well as in $\mathbf{x}'$. The delta function is defined such that

$$\int \delta\left(\mathbf{y}\right) d^3 y = 1 \tag{4.4}$$

but now its argument is $\mathbf{y} = \mathbf{x}' - \mathbf{r}\left(t'\right)$. We need to relate $d^3 y$ to $d^3 x'$ for the integral we want. Consider the gradient of one component:

$$
\begin{aligned}
\nabla' y_i &= \nabla'\left(x_i' - r_i\left(t'\right)\right) = \nabla'[\![x_i' - r_i]\!] \\
&= [\![\nabla'\left(x_i' - r_i\right)]\!] + \frac{\mathbf{x} - \mathbf{x}'}{c|\mathbf{x} - \mathbf{x}'|}\left[\!\!\left[\frac{\partial}{\partial t}\left(x_i' - r_i\right)\right]\!\!\right] \\
&= [\![\nabla' x_i']\!] + \frac{1}{c}\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}\left[\!\!\left[-\frac{\partial r_i}{\partial t}\right]\!\!\right]
\end{aligned}
\tag{4.5}
$$

[since r_i is a function of t but not $\mathbf{x}'$ directly.] Choose axes such that $\mathbf{x}' = \left(x_1', x_2', x_3'\right)$ with component 1 in the $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ direction. Then the second term is present only for the $\mathbf{x}_1$ component not the other two (because they are $\perp$ to $\mathbf{x} - \mathbf{x}'$). Also $\left(\nabla' x_i'\right) = \delta_{ij}$. [i.e. 1 iff $i = j$]. Thus

$$
\begin{aligned}
\frac{\partial y_1}{\partial x_1'} &= 1 + \frac{1}{c}\left[\!\!\left[-\frac{\partial r_1}{\partial t}\right]\!\!\right] \\
\frac{\partial y_2}{\partial x_2'} &= 1 \quad ; \quad \frac{\partial y_3}{\partial x_3'} = 1 \quad .
\end{aligned}
\tag{4.6}
$$

Consequently

$$
\begin{aligned}
d^3 y &= dy_1 dy_2 dy_3 = \left(1 - \frac{1}{c}\left[\!\!\left[\frac{\partial r_1}{\partial t}\right]\!\!\right]\right) dx_1' dx_2' dx_3' \\
&= \left[\!\!\left[1 - \frac{1}{c}\frac{\mathbf{R}}{R}\cdot\frac{\partial \mathbf{r}}{\partial t}\right]\!\!\right] d^3 x'
\end{aligned}
\tag{4.7}
$$

Let's write

$$\kappa \equiv 1 - \frac{1}{c}\frac{\mathbf{R}}{R}\cdot\frac{\partial \mathbf{r}}{\partial t} = 1 - \frac{1}{c}\hat{\mathbf{R}}.\mathbf{v} \tag{4.8}$$

Then

$$\int \delta\left(\mathbf{y}\right) d^3 \mathbf{x}' = \int \delta\left(\mathbf{y}\right)\frac{d^3 y}{[\![\kappa]\!]} = \frac{1}{[\![\kappa]\!]} \tag{4.9}$$

And finally

$$\phi\left(\mathbf{x}, t\right) = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa R}\right]\!\!\right] \tag{4.10}$$

By exactly the same process we can obtain the correct value for each component of $\mathbf{A}$ and in total

$$\mathbf{A}\left(\mathbf{x}, t\right) = \frac{\mu_0 q}{4\pi}\left[\!\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\!\right] \quad . \tag{4.11}$$

$(\mathbf{v} = \partial\mathbf{r}/\partial t)$, $\mathbf{j} = q\mathbf{v}\delta$. These expressions are called the "Liénard-Wiechert" potentials of a moving point charge. Since the κ correction factor is so important and the scientific literature is strewn with papers that get it wrong, let's obtain the result graphically. The retarded
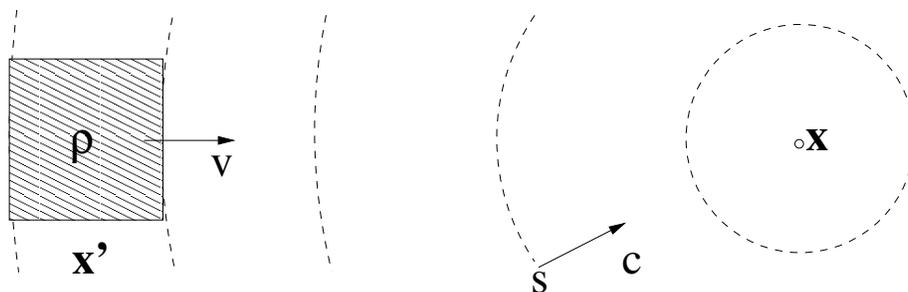
Figure 4.2: Integral of charge density over a square-shaped moving charge at retarded time.

integral $\int [\![\rho]\!]\, d^3x'$ can be viewed as composed of contributions from a spherical surface S which sweeps inward towards the observation (field) point $\mathbf{x}$, at the speed of light, arriving at time t. The charge that we integrate is the value of ρ when the surface S sweeps past. If we are dealing with a localized charge density, such as illustrated, the surface can be approximated as planar at the charge. If the charge region is moving rigidly at speed v towards $\mathbf{x}$, then its influence or contribution to the integral is increased because by the time the surface S has swept from front to back, the charge has moved. Consequently, the volume of the contribution (in x') is larger by the ratio $\frac{L'}{L}$ of the $\frac{\text{additional volume} + \text{charge volume}}{\text{charge volume}}$ How much is this? When does S reach 'front' of charge? At the moment S reaches the front,
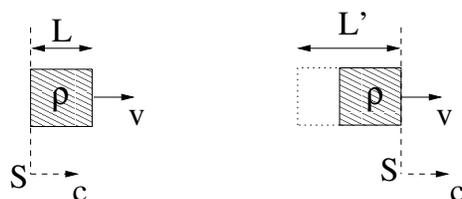
Figure 4.3: Snapshots as the integration surface, S, crosses the back and the front of the charge.

$$L' = c\Delta t = v\Delta t + L \tag{4.12}$$

So

$$\Delta t = \frac{L}{(c-v)} \qquad \text{and} \qquad L' = \frac{c}{c-v}L = \frac{1}{1-\frac{v}{c}}L \tag{4.13}$$

Thus

$$\frac{L'}{L} = \frac{1}{1-\frac{v}{c}} = \frac{1}{\kappa} \tag{4.14}$$

as before. Notice that transverse velocity does nothing, and that approximations implicit in taking S to be planar become exact for a point charge, with spatial extent $\rightarrow 0$. The

quantity κ can also be seen to relate intervals of time, dt, to the corresponding retarded time intervals, dt'.

$$t' = t - \frac{R'}{c} \qquad \text{or} \qquad t = t' + \frac{R'}{c} \tag{4.15}$$

So

$$\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR'}{dt'} \qquad . \tag{4.16}$$

But

$$\begin{aligned}
\frac{dR'}{dt'} &= \frac{d}{dt'}|\mathbf{x} - \mathbf{r}'| = \frac{d}{dt'}\left\{(\mathbf{x} - \mathbf{r}').(\mathbf{x} - \mathbf{r}')\right\}^{1/2} \\
&= \frac{-\mathbf{v}.(\mathbf{x} - \mathbf{r}')}{R'} = -\frac{\mathbf{v}.\mathbf{R}'}{R'}
\end{aligned} \tag{4.17}$$

Hence

$$\frac{dt}{dt'} = 1 - \frac{\mathbf{v}.\mathbf{R}'}{cR'} = \kappa' = [\![\kappa]\!] \tag{4.18}$$

Strictly speaking, it is the value κ' at retarded time, when the surface S passes the particle, that is required here if $\mathbf{v}$ is changing.

4.2   Potential of a Point Charge in Uniform Motion [1]
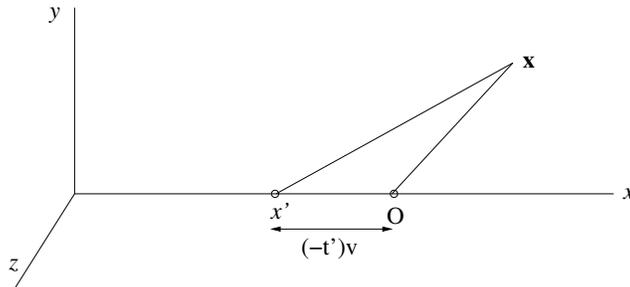
Figure 4.4: Coordinates of a uniformly moving charge at $\mathbf{r}(t) = (vt, 0, 0)$.

An important special case is when $\mathbf{v} = \dot{\mathbf{r}} = \text{const}$. From the retarded solution Lorentz derived his transformation, which is the basis of special relativity. Take axes such that $\mathbf{v} = v\hat{\mathbf{x}}$. We need to calculate the potential at $\mathbf{x} = (x, y, z)$ and we'll suppose that the particle is at the origin at time of interest, $(t = 0)$. Tricky part is just to calculate the retarded time t' and position $\mathbf{x}'$, which are related by the expression $(-t')v = -x'$. The retarded time is just that time it takes light to travel the distance R'; so

$$c^2\left(-t'\right)^2 = \frac{c^2}{v^2}x'^2 = R'^2 = (x - x')^2 + y^2 + z^2 \qquad . \tag{4.19}$$

---

[1][Feynman 21-6]

Gather terms in the x' form of the equation:

$$x'^2 \left(\frac{c^2}{v^2} - 1 \right) + 2xx' - \left(x^2 + y^2 + z^2 \right) = 0. \tag{4.20}$$

Solution of quadratic in x':

$$\left(\frac{c^2}{v^2} - 1 \right) x' = -x \pm \sqrt{x^2 + (x^2 + y^2 + z^2) \left(\frac{c^2}{v^2} - 1 \right)} \tag{4.21}$$

(where the - sign must be taken). And so

$$R' = -\frac{c}{v}x' = \frac{c}{v} \frac{x + \sqrt{x^2 + (x^2 + y^2 + z^2)\left(\frac{c^2}{v^2} - 1 \right)}}{\frac{c^2}{v^2} - 1}. \tag{4.22}$$

We also need the retarded value of κ i.e. $1 - (\mathbf{R'}/R').(\mathbf{v}/c)$.

$$\kappa' = 1 - \frac{v}{c} \cdot \frac{x - x'}{R'} \tag{4.23}$$

and

$$\kappa' R' = R' - \frac{v}{c}(x - x') = R' - \frac{v}{c}\left(x + \frac{v}{c}R' \right) = \left(1 - \frac{v^2}{c^2} \right) R' - \frac{v}{c}x \tag{4.24}$$

Substituting for R' we get

$$
\begin{aligned}
\kappa' R' &= \frac{v}{c}\left\{ x + \sqrt{x^2 + (x^2 + y^2 + z^2)\left(\frac{c^2}{v^2} - 1 \right)} - x \right\} \\
&= \frac{v}{c}\sqrt{x^2 + (x^2 + y^2 + z^2)\left(\frac{c^2}{v^2} - 1 \right)} \\
&= \sqrt{\frac{x^2}{1 - \frac{v^2}{c^2}} + y^2 + z^2}\sqrt{1 - \frac{v^2}{c^2}}
\end{aligned}
\tag{4.25}
$$

This is the value at time $t = 0$. At any other time t, the particle is at the position $x = vt$ instead of at the origin, $x = 0$. Our formula was developed for the particle at the origin. So to use it we must move the origin to $x = vt$, which means we simply have to replace x in this formula with $x - vt$. So finally, substituting the general result for $\kappa' R'$ into the Liénard-Wiechert formula we get

$$\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left[\!\left[\frac{1}{\kappa R} \right]\!\right] = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}} \right)^2 + y^2 + z^2}} \tag{4.26}$$

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See how we have the beginnings of relativity. We get electromagnetic potential dependence on spatial coordinates that can only be consistent with the formula in the frame of reference in which the particle is at rest:

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} \tag{4.27}$$

if coordinates transform as

$$x_1 = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \qquad y_1 = y \qquad , \qquad z_1 = z \quad . \tag{4.28}$$

This is the (spatial part of the) Lorentz transformation, incorporating the Fitzgerald contraction in the direction of motion. Now we also need to recognize there is a vector potential

$$\mathbf{A} = \frac{\mu_0 q}{4\pi} \left[\!\!\left[\frac{\mathbf{v}}{\kappa R} \right]\!\!\right] = \frac{\mu_0 q}{4\pi} \frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}} \sqrt{\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}} \right)^2 + y^2 + z^2}} \tag{4.29}$$

So the electric field has both contributions:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad . \tag{4.30}$$

To evaluate these, denote by R_1 the quantity in the denominator of ϕ and $\mathbf{A}$:

$$R_1 \equiv \sqrt{\left(\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^2 + y^2 + z^2} \quad . \tag{4.31}$$

[Note that this is <u>not</u> R', the retarded radius]. Its derivatives are

$$\frac{\partial R_1}{\partial x} = \frac{x - vt}{1 - \frac{v^2}{c^2}} \frac{1}{R_1} \quad ; \quad \frac{\partial R_1}{\partial y} = \frac{y}{R_1} \quad ; \quad \frac{\partial R_1}{\partial z} = \frac{z}{R_1} \quad ; \quad \frac{\partial R_1}{\partial t} = \frac{-v(x - vt)}{\left(1 - \frac{v^2}{c^2}\right) R_1} \quad . \tag{4.32}$$

Consequently

$$\nabla \frac{1}{R_1} = \frac{-1}{R_1^2} \nabla R_1 = -\frac{1}{R_1^3} \left(\frac{x - vt}{1 - v^2/c^2}, y, z \right) \tag{4.33}$$

giving

$$-\nabla\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{1}{R_1^3} \left(\frac{x - vt}{1 - v^2/c^2}, y, z \right) \tag{4.34}$$

and

$$-\frac{\partial \mathbf{A}}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - v^2/c^2}} \frac{1}{R_1^3} \left(\frac{(-v^2/c^2)(x - vt)}{(1 - v^2/c^2)}, 0, 0 \right) \tag{4.35}$$

so

$$\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1 - v^2/c^2}}\frac{1}{R_1^3}\left(x - vt\,, y, z\right) \quad . \tag{4.36}$$

This is a remarkable result. It shows that despite the fact that contributions to E arise from the <u>retarded</u> position of the particle, the direction of $\mathbf{E}$ is actually radially outward from the instantaneous (i.e. <u>non</u> retarded) position. The $\mathbf{E}$ field at $t = 0$ is along the radius vector (x, y, z). The electric field is not just the same as for a stationary charge. The field is not
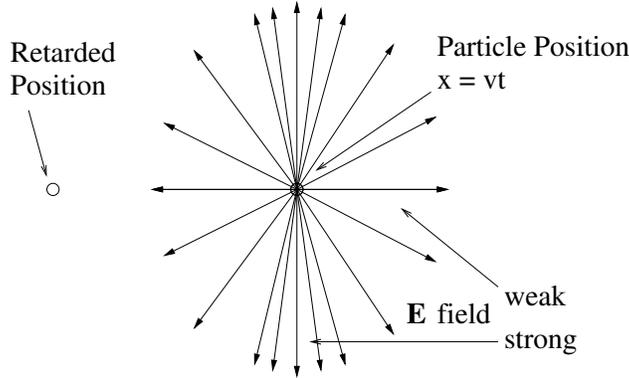
Figure 4.5: Electric field lines of a charge in uniform motion point outward from the instantaneous (not retarded) position but the field strength is not symmetric.

spherically symmetric, since it is proportional to

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}\left(\left(\frac{x - vt}{\sqrt{(1-v^2/c^2)}}\right)^2 + y^2 + z^2\right)^{3/2}} \tag{4.37}$$

which makes it stronger in the perpendicular direction and weaker in the parallel direction.

The magnetic field may be obtained from $\mathbf{B} = \nabla \wedge \mathbf{A}$ by recognizing $\nabla \wedge (f\mathbf{v}) = -\mathbf{v} \wedge \nabla f$, if $\mathbf{v}$ is constant. Hence, using $\mathbf{A} = \mathbf{v}\phi/c^2$,

$$\mathbf{B} = -\frac{\mathbf{v}}{c^2} \wedge \nabla\phi = \frac{\mathbf{v}}{c^2} \wedge \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \frac{1}{c^2}\mathbf{v} \wedge \mathbf{E} \quad . \tag{4.38}$$

[The latter form uses the fact that ($\mathbf{A}$ and) $\frac{\partial \mathbf{A}}{\partial t}$ are parallel to $\mathbf{v}$ so $\mathbf{v} \wedge \frac{\partial \mathbf{A}}{\partial t} = 0$]. This expression for the magnetic field can also be rewritten, by noticing that $\mathbf{E}$ is in the direction of $\mathbf{R}$, $\mathbf{R}' \wedge \mathbf{R} = (t - t')\mathbf{v} \wedge \mathbf{R}$ and $t - t' = R'/c$; so $\mathbf{v} \wedge \mathbf{E} = (\mathbf{R}'c/R') \wedge \mathbf{E}$. To summarize:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x} - \mathbf{v}t}{\sqrt{1 - \frac{v^2}{c^2}}\left(\left(\frac{x - vt}{\sqrt{(1-v^2/c^2)}}\right)^2 + y^2 + z^2\right)^{3/2}} \tag{4.39}$$

and

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v} \wedge \mathbf{E} = \frac{1}{cR'}\mathbf{R}' \wedge \mathbf{E} \quad . \tag{4.40}$$

A helpful way to think of the result that the electric field is still radial but with a non-spherically-symmetric distribution, is to think about what happens to the field lines when viewed in the lab frame of reference [components (x, y, z)] compared with a frame of reference in which the particle is at rest [components (x_1, y_1, z_1)]. It turns out that the electric field we have calculated is exactly that which would be obtained by assuming that the spherically symmetric distribution of field-lines in the rest-frame is simply compressed together with the rest of space in the x-direction through the coordinate transform of eq 4.28. This contraction is illustrated in figure 4.6.
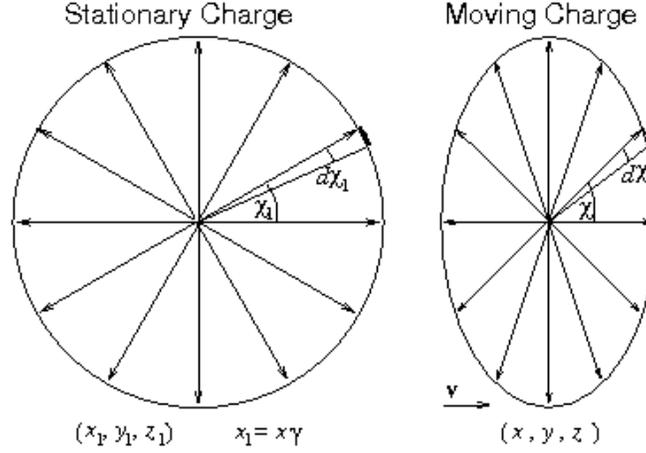
Figure 4.6: Contraction of space which gives the electric field-line distribution of a moving charge.

For a purely geometric compression in one dimension like this, the angles between the direction of $\mathbf{R}$ and $\mathbf{v}$ (for the two cases) are related by

$$\tan \chi_1 = y_1/x_1 = y/(\gamma x) = (1/\gamma) \tan \chi \quad , \tag{4.41}$$

where $\gamma = 1/\sqrt{(1 - v^2/c^2)}$. Consequently

$$\cos \chi_1 = \frac{1}{\sqrt{1 + \tan^2 \chi_1}} = \frac{1}{\sqrt{1 + (1/\gamma)^2 \tan^2 \chi}} \quad . \tag{4.42}$$

Now the element of solid angle corresponding to an angle increment $d\chi$ is $\mathbf{d\Omega} = 2\pi \sin \chi \, d\chi$ and

$$
\begin{aligned}
\sin \chi_1 \, d\chi_1 = -d(\cos \chi_1) &= -d \left(\frac{\gamma}{\sqrt{\gamma^2 + \tan^2 \chi}} \right) = \frac{\gamma \tan \chi \, \sec^2 \chi}{(\gamma^2 + \tan^2 \chi)^{3/2}} \, d\chi \\
&= \frac{\gamma}{(\gamma^2 \cos^2 \chi + \sin^2 \chi)^{3/2}} \sin \chi \, d\chi \quad .
\end{aligned}
\tag{4.43}
$$

So the relationship between corresponding solid-angles is

$$
\begin{aligned}
d\Omega_1 &= \frac{\gamma}{(\gamma^2 \cos^2 \chi + \sin^2 \chi)^{3/2}} \, d\Omega \\
&= \frac{\gamma R^3}{(\gamma^2 x^2 + y^2 + z^2)^{3/2}} \, d\Omega \\
&= \frac{\gamma R^3}{R_1^3} \, d\Omega \quad ,
\end{aligned}
\tag{4.44}
$$

where $R = x^2 + y^2 + z^2$. Therefore if the field-lines are compressed in this purely geometrical way, the number of field-lines per unit solid angle, which is proportional to the electric field intensity, in the lab-frame is equal to the value in the rest-frame times the factor $d\Omega_1/d\Omega = \gamma R^3/R_1^3$. Thus the geometric compression would lead to an electric field:

$$
\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} \frac{\gamma R^3}{R_1^3} \quad .
\tag{4.45}
$$

This is precisely what we calculated directly from the equations of the fields. In other words, we can regard the non-symmetric electric field of eq 4.39 as arising from a compression of space corresponding to the Lorentz transformation (eq 4.28).

We are not here invoking the Lorentz transformation based on an understanding of special relativity. In fact the opposite is the historic situation. Lorentz's transform was part of the prior basis for the discovery of relativity. See Jackson 1998 pp. 514-518 for a discussion of electromagnetism as the historic foundation of relativity. Maxwell's equations are already fully relativistic. They don't need to be corrected for relativistic effects, the way Newton's laws require correction for example. Of course the point is stronger than that: Maxwell's equations can only be consistent when special relativity applies (i.e. Lorentz, not Galilean transformations). We don't have time to cover relativity but we don't have to make a special point of it since EM equations already are relativisitc.

4.3   Fields of a Generally-Moving Charge

The Lienard Wiechert potentials give the general potential solution. From them we can obtain the general $\mathbf{E}$ and $\mathbf{B}$ fields from a particle moving with arbitrary velocity: not just uniform $\mathbf{v}$. Since both potentials and fields depend only on the values at retarded time, our calculation will be almost the same as for the uniform motion with the exception that we must use the value of $\mathbf{v}$ at that retarded time *and* we must account for possible time-derivatives of $\mathbf{v}$. Our derivations of ϕ and $\mathbf{A}$ go through exactly as before except that the origin of coordinates is at a point $\mathbf{x}' + \mathbf{v}'t'$ along the *projected* path of the particle if it were to continue past the retarded time with constant speed $\mathbf{v}'$. [Here we are putting prime on $\mathbf{v}$ to remind that it is the retarded value we require.]

$$
\phi\left(\mathbf{x}, t\right) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x - v't) + (y^2 + z^2)\left(1 - \frac{v'^2}{c^2}\right)}}
\tag{4.46}
$$

$$\mathbf{A}\left(\mathbf{x},t\right) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}'}{\sqrt{(x-v't)+(y^2+z^2)\left(1-\frac{v'^2}{c^2}\right)}} \tag{4.47}$$

Now we need to get the fields by differentiation. We get exactly the same terms as before plus extra terms arising from the time derivative of $\mathbf{v}$. We <u>could</u> do this directly by taking into account all the contributions. Instead, let's do a vector calculation starting with the Lienard-Wiechert forms:

$$\phi = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa R}\right]\!\!\right] \quad ; \quad \mathbf{A} = \frac{q}{4\pi\epsilon_0 c^2}\left[\!\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\!\right] \quad . \tag{4.48}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} = \frac{q}{4\pi\epsilon_0}\left\{-\nabla\left[\!\!\left[\frac{1}{\kappa R}\right]\!\!\right] - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\!\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\!\right]\right\} \tag{4.49}$$

Again, extreme care must be taken with the differentials. For any function $f(\mathbf{x},t)$,

$$\nabla[\![f]\!] = \nabla f\left(\mathbf{x},t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

$$= [\![\nabla f]\!] - \left[\!\!\left[\frac{\partial f}{\partial t}\right]\!\!\right]\frac{1}{c}\nabla[\![|\mathbf{x}-\mathbf{x}'|]\!] \tag{4.50}$$

This is <u>not</u> the same situation as we had before. There we had ∇' i.e. gradient with respect to retarded position, $\mathbf{x}'$, keeping $\mathbf{x}$ and t fixed. Here we are talking about gradient w.r.t. $\mathbf{x}$ keeping t fixed. Apply the above equation to the function $|\mathbf{x}-\mathbf{x}'|$ which is, strictly speaking, $[\![|\mathbf{x}-\mathbf{r}|]\!]$ or $[\![R]\!]$. We get

$$\nabla[\![|\mathbf{x}-\mathbf{r}|]\!] = [\![\nabla|\mathbf{x}-\mathbf{r}|]\!] - \left[\!\!\left[\frac{\partial}{\partial t}|\mathbf{x}-\mathbf{r}|\right]\!\!\right]\frac{1}{c}\nabla[\![|\mathbf{x}-\mathbf{r}|]\!] \tag{4.51}$$

i.e.

$$\left[\!\!\left[\left(1-\frac{\mathbf{R}'}{R'}\cdot\frac{\mathbf{v}}{c}\right)\right]\!\!\right]\nabla[\![|\mathbf{x}-\mathbf{r}|]\!] = [\![\nabla|\mathbf{x}-\mathbf{r}|]\!] \tag{4.52}$$

So

$$\nabla[\![R]\!] = \left[\!\!\left[\frac{1}{\kappa}\nabla R\right]\!\!\right] = \left[\!\!\left[\frac{\mathbf{R}}{\kappa R}\right]\!\!\right] \quad . \tag{4.53}$$

Then returning to the general identity (4.50), substitute for $\nabla[\![R]\!]$ to find:

$$\nabla[\![f]\!] = [\![\nabla f]\!] - \left[\!\!\left[\frac{\partial f}{\partial t}\right]\!\!\right]\frac{1}{c}\left[\!\!\left[\frac{\mathbf{R}}{\kappa R}\right]\!\!\right] \tag{4.54}$$

or finally

$$\nabla[\![f]\!] = \left[\!\!\left[\nabla f - \frac{\mathbf{R}}{c\kappa R}\frac{\partial f}{\partial t}\right]\!\!\right] \quad . \tag{4.55}$$

For the time derivative, we proceed more directly, recognizing that $\frac{\partial f}{\partial t}$ means differential of f with respect to its second argument, but $\frac{\partial}{\partial t}[\![f]\!]$ means differential with respect to t of a function whose second (time-like) argument is t'. Thus

$$\frac{\partial}{\partial t}[\![f]\!] = \frac{\partial}{\partial t}f(\mathbf{x}, t') = \frac{dt'}{dt}\frac{\partial}{\partial t'}f(\mathbf{x}, t') = \frac{1}{\kappa'}\left[\!\!\left[\frac{\partial f}{\partial t}\right]\!\!\right] \tag{4.56}$$

using equation (4.18), $dt/dt' = \kappa' = [\![\kappa]\!]$. Therefore we have generally

$$\frac{\partial}{\partial t}[\![f]\!] = \left[\!\!\left[\frac{1}{\kappa}\frac{\partial f}{\partial t}\right]\!\!\right] \quad . \tag{4.57}$$

Ok, now we have the general derivative tools to evaluate $\mathbf{E}$. It becomes

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[-\nabla\left(\frac{1}{\kappa R}\right) + \frac{\mathbf{R}}{c\kappa R}\frac{\partial}{\partial t}\left(\frac{1}{\kappa R}\right) - \frac{1}{c^2\kappa}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right)\right]\!\!\right] \quad . \tag{4.58}$$

With everything inside the retardation operator, it is safe to proceed with algebra using straightforward derivatives with respect to the spatial and temporal arguments of the functions, and the fact that $\partial\mathbf{R}/\partial t = -\partial\mathbf{r}/\partial t = -\mathbf{v}$. In particular,

$$\kappa R = R - \frac{\mathbf{R}.\mathbf{v}}{c} \tag{4.59}$$

$$\nabla(\kappa R) = \frac{\mathbf{R}}{|R|} - \frac{\mathbf{v}}{c} \tag{4.60}$$

$$\frac{\partial}{\partial t}(\kappa R) = \frac{\partial R}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{R}}{\partial t}.\mathbf{v} - \frac{\mathbf{R}}{c}.\frac{\partial\mathbf{v}}{\partial t}$$

$$= \dot{R} + \frac{v^2}{c} - \frac{1}{c}\mathbf{R}.\dot{\mathbf{v}} \tag{4.61}$$

where dot denotes $\frac{\partial}{\partial t}$. The terms in $\mathbf{E}$ are then

$$-\nabla\left(\frac{1}{\kappa R}\right) = \frac{1}{\kappa^2 R^2}\nabla(\kappa R) = \frac{1}{\kappa^2 R^2}\left(\frac{\mathbf{R}}{R} - \frac{\mathbf{v}}{c}\right) \tag{4.62}$$

$$\frac{1}{c\kappa}\frac{\mathbf{R}}{R}\frac{\partial}{\partial t}\left(\frac{1}{\kappa R}\right) = -\frac{1}{c\kappa^3 R^3}\mathbf{R}\frac{\partial}{\partial t}(\kappa R) = -\frac{1}{c\kappa^3 R^3}\mathbf{R}\left(\dot{R} + \frac{v^2}{c} - \frac{1}{c}\mathbf{R}.\dot{\mathbf{v}}\right) \tag{4.63}$$

$$-\frac{1}{c^2\kappa}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right) = \frac{1}{c^2\kappa^3 R^2}\left(\dot{R} + \frac{v^2}{c} - \frac{1}{c}\mathbf{R}.\dot{\mathbf{v}}\right)\mathbf{v} - \frac{1}{c^2\kappa^2 R}\dot{\mathbf{v}} \tag{4.64}$$

Gathering terms together, denoting $\hat{\mathbf{R}} = \frac{\mathbf{R}}{R}$, and using $\dot{R} = -\mathbf{v}.\mathbf{R}$, we get

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa^3 R^2}\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) - \frac{1}{c\kappa^3 R}\left\{\kappa\frac{\dot{\mathbf{v}}}{c} - \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\left(\hat{\mathbf{R}}.\frac{\dot{\mathbf{v}}}{c}\right)\right\}\right]\!\!\right] \tag{4.65}$$

or alternatively, using vector triple product identities,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa^3 R^2}\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c\kappa^3 R}\hat{\mathbf{R}}\wedge\left[\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\wedge\frac{\dot{\mathbf{v}}}{c}\right]\right]\!\!\right] . \tag{4.66}$$

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The magnetic field is $\mathbf{B} = \nabla \wedge \mathbf{A}$ which becomes

$$
\begin{aligned}
\mathbf{B} &= \frac{q}{4\pi\epsilon_0 c^2} \left\{ \nabla \wedge \left[\kern-0.25em\left[\frac{\mathbf{v}}{\kappa R} \right]\kern-0.25em\right] \right\} \\
&= \frac{q}{4\pi\epsilon_0 c^2} \left[\kern-0.25em\left[\nabla \wedge \left(\frac{\mathbf{v}}{\kappa R} \right) - \frac{\mathbf{R}}{c\kappa R} \wedge \frac{\partial}{\partial t} \left(\frac{\mathbf{v}}{\kappa R} \right) \right]\kern-0.25em\right] \quad ,
\end{aligned}
\tag{4.67}
$$

by an identity directly analogous to the one we showed for gradient. Also, using eq 4.62 and noting that terms of the form $\mathbf{v} \wedge \mathbf{v}$ and $\hat{\mathbf{R}} \wedge \hat{\mathbf{R}}$ can be inserted or removed at will since they are always zero we have

$$
\begin{aligned}
\nabla \wedge \left(\frac{\mathbf{v}}{\kappa R} \right) &= -\mathbf{v} \wedge \nabla \left(\frac{1}{\kappa R} \right) = \frac{1}{\kappa^2 R^2} \mathbf{v} \wedge \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c} \right) \\
&= -\frac{1}{\kappa^2 R^2} c\hat{\mathbf{R}} \wedge \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c} \right) = -c\hat{\mathbf{R}} \wedge \nabla \left(\frac{1}{\kappa R} \right) \quad .
\end{aligned}
\tag{4.68}
$$

Hence

$$
\mathbf{B} = \frac{q}{4\pi\epsilon_0 c^2} \left[\kern-0.25em\left[-c\hat{\mathbf{R}} \wedge \nabla \left(\frac{1}{\kappa R} \right) - \hat{\mathbf{R}} \wedge \frac{1}{c\kappa} \frac{\partial}{\partial t} \left(\frac{\mathbf{v}}{\kappa R} \right) \right]\kern-0.25em\right] = \frac{1}{c} \left[\kern-0.25em\left[\hat{\mathbf{R}} \right]\kern-0.25em\right] \wedge \mathbf{E} \quad ,
\tag{4.69}
$$

by comparison with our expression (4.58) for $\mathbf{E}$.

Summarizing our results, the fields due to a point charge q moving with variable velocity $\mathbf{v}$ such that the radius vector from the charge to the field-point is $\mathbf{R}$ may be expressed using $\kappa \equiv 1 - \hat{\mathbf{R}}.\mathbf{v}/c$ as:

$$
\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0} \left[\kern-0.25em\left[\frac{1}{\kappa^3 R^2} \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c} \right) \left(1 - \frac{v^2}{c^2} \right) + \frac{1}{c\kappa^3 R} \hat{\mathbf{R}} \wedge \left[\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c} \right) \wedge \frac{\dot{\mathbf{v}}}{c} \right] \right]\kern-0.25em\right] \tag{4.70} \\
\mathbf{B} &= \frac{1}{c} \left[\kern-0.25em\left[\hat{\mathbf{R}} \right]\kern-0.25em\right] \wedge \mathbf{E} \quad . \tag{4.71}
\end{aligned}
$$

There are several different forms of these expressions, useful to illustrate different aspects of the fields of a moving point charge. See Jackson and Feynman for discussion of some of these.

4.4 Radiation from Moving Charges

4.4.1 Near Field and Radiation Terms

The form for $\mathbf{E}$ that we obtained was exhibited in a way that had 2 separate terms. The first of those terms does not contain $\dot{\mathbf{v}}$ while the second is proportional to $\dot{\mathbf{v}}$. Therefore the first term is exactly what would be obtained for uniform motion $\dot{\mathbf{v}} = 0$ (although this is not obvious when comparing with our earlier formula expressed in coordinates). Also, everything inside the brackets is dimensionless $(\hat{R}, \frac{\mathbf{v}}{c})$ *except* $\frac{1}{\kappa^3 R^2}$ and $\frac{1}{c\kappa^3 R} \frac{\dot{\mathbf{v}}}{c}$. These factors decide the

behaviour of their respective terms at large field-point distances, R. The 'static' (constant $\mathbf{v}$) term is $\propto \frac{1}{R^2}$ but the $\dot{\mathbf{v}}$ term is $\propto \frac{1}{R}$. Consequently, the Poynting vector is

$$\mathbf{E} \wedge \mathbf{H} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E} \wedge \left(\left[\!\left[\hat{\mathbf{R}}\right]\!\right] \wedge \mathbf{E}\right) \propto \frac{1}{R^4} \qquad \text{or} \qquad \frac{1}{R^2} \tag{4.72}$$

respectively. If we ask about the total EM power flux across a spherical surface far from the charge, that value scales like the surface area $4\pi R^2$ tims $\mathbf{E} \wedge \mathbf{H}$. Thus power flux $\propto \frac{1}{R^2}$ for the constant v term, and $\propto 1$ for the $\dot{\mathbf{v}}$ term. We see then, that the constant-v term gives rise to vanishingly small power flux far from the charge but the $\dot{\mathbf{v}}$ term gives rise to finite power flux even at infinity. This distinction requires us to regard these two terms as the "near field" term:

$$E \propto \frac{1}{R^2}v \tag{4.73}$$

and "radiation" term:

$$E \propto \frac{1}{R}\dot{v} \quad . \tag{4.74}$$

A charged particle *radiates* only if it *accelerates.*

4.4.2 Radiation into a Specific Solid-angle

Having identified just the $1/R$ term as the radiation term, we will drop the other, near field, term from consideration. Imagine, then, a sphere of radius R surrounding the retarded position of the particle. The Poynting vector of the radiation term there is

$$\begin{aligned}
\mathbf{E} \wedge \mathbf{B}/\mu_0 &= \mathbf{E} \wedge \left(\left[\!\left[\hat{\mathbf{R}}\right]\!\right] \wedge \mathbf{E}\right)/c\mu_0 = \left(E^2/c\mu_0\right)\left[\!\left[\hat{\mathbf{R}}\right]\!\right] - \left(\mathbf{E}.\left[\!\left[\hat{\mathbf{R}}\right]\!\right]\right)\mathbf{E}/c\mu_0 \\
&= \frac{1}{c\mu_0}E^2\left[\!\left[\hat{\mathbf{R}}\right]\!\right] = c\epsilon_0 E^2\left[\!\left[\hat{\mathbf{R}}\right]\!\right] \quad ,
\end{aligned} \tag{4.75}$$

where the last two forms recognize that the radiation term has $\mathbf{E}$ perpendicular to $\left[\!\left[\hat{\mathbf{R}}\right]\!\right]$. Radiated energy thus crosses the sphere, normal to its surface with a local intensity (energy/unit area/unit time) $E^2/c\mu_0$, with E given by the second term of eq (4.70). One is very often interested in the power radiated per unit solid angle, Ω_s, subtended by the area at the point of radiation. By definition of solid angle, a small area of the sphere, A, subtends a solid angle A/R^2. Consequently the power per unit solid angle is $R^2E^2/c\mu_0$. The extra term R^2 cancels the R^2 occurring in E^2, leaving an expression independent of the radius, R, of the sphere. By convention we can write the power per unit solid angle using the notation

$$\frac{dP}{d\Omega_s} = \frac{R^2 E^2}{c\mu_0} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\left|\left[\!\left[\frac{1}{\kappa^3}\hat{\mathbf{R}} \wedge \left\{\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right) \wedge \frac{\dot{\mathbf{v}}}{c}\right\}\right]\!\right]\right|^2 \tag{4.76}$$

4.4.3 Radiation from Non-relativistic Particles: Dipole Approximation

Considerable algebraic simplifications occur when $v/c \ll 1$ and so we can approximate $(\hat{\mathbf{R}} - \mathbf{v}/c) \simeq \hat{\mathbf{R}}$, and $\kappa = 1$. Then

$$\frac{dP}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} |\hat{\mathbf{R}} \wedge \left(\hat{\mathbf{R}} \wedge \frac{\dot{\mathbf{v}}}{c}\right)|^2 = \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} \left(\frac{\dot{\mathbf{v}}}{c}\right)^2 \sin^2\alpha \tag{4.77}$$

where α is the angle between $\hat{\mathbf{R}}$, the direction of the solid angle (propagation), and $\dot{\mathbf{v}}$, the acceleration. An integration of the total radiated power over the entire sphere (all solid-angles) can readily be done. Taking the direction of $\dot{\mathbf{v}}$ to be the polar direction, the integral is such that

$$d\Omega_s = 2\pi \sin\alpha \, d\alpha \quad . \tag{4.78}$$

So, noting that

$$\int \sin^2\alpha \, 2\pi \sin\alpha \, d\alpha = 2\pi \int \left(1 - \cos^2\alpha\right) \sin\alpha \, d\alpha = 2\pi \left[-\cos\alpha + \frac{1}{3}\cos^3\alpha\right]_0^\pi = \frac{8\pi}{3} \quad , \tag{4.79}$$

we get

$$P = \int \frac{dP}{d\Omega_s} d\Omega_s = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \left(\frac{\dot{v}}{c}\right)^2 \quad . \tag{4.80}$$

This expression for the total radiation from a non-relativistic accelerated charge is known as Larmor's formula. The non-relativistic expressions for P and $dP/d\Omega_s$ are often referred to as the "dipole approximation" because they are exactly what is obtained for the radiation from a stationary oscillating dipole electric distribution when the electric dipole moment $\mathbf{p}$, is such that

$$\ddot{\mathbf{p}} = q\dot{\mathbf{v}} \quad . \tag{4.81}$$

Thus this radiation pattern and intensity is what is obtained also from dipole antennas that are much smaller than the radiation wavelength.

4.5 Radiation from Relativistic Particles

The general expression for radiation by an accelerated particle, without invoking approximations requiring $v \ll c$, is given by eq (4.76). However an important distinction must be drawn in discussions of energy per unit time between expressions based on time-at-field-point, t, such as eq (4.76), and expressions referring to time-at-particle, retarded time t'. If we want to know how much energy a particle is radiating per unit time-at-particle, which is what we do want if, for example, we want to calculate how rapidly the particle is losing energy, or indeed if we want to calculate the total energy radiated per unit volume by adding up the energy radiated by all the particles in that volume, then we must multiply expressions for energy per time-at-field-point by the ratio $dt/dt' = \kappa$. This conversion lowers the power of

κ in the denominator by one. We shall work henceforth with such expressions of energy per unit time-at-particle and will indicate this by a prime on the power: P'. Even so, we still have a factor κ^5 in the denominator of $dP'/d\Omega_s$. This factor is the most important effect. Since $\kappa = 1 - \hat{\mathbf{R}}.\mathbf{v}/c = 1 - \beta\cos\theta$, when we are dealing with particles moving near the speed of light, κ becomes extremely small when $\theta \simeq 0$, that is for radiation in the direction along the particle's velocity. As a result, the radiation is greatly enhanced in this forward direction, an effect that is sometimes called the relativistic "headlight" effect.

4.5.1 Acceleration Parallel to v

The simplest case algebraically is when $\mathbf{v}$ and $\dot{\mathbf{v}}$ are parallel. The radiation is then rotationally symmetric about this direction. The κ factor is then the only difference from the dipole formula eq (4.77):

$$\frac{dP'}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0} \frac{\dot{v}^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \quad . \tag{4.82}$$

The radiation in the exactly forward direction $\theta = 0$ is zero because of the $\sin^2\theta$ term. The
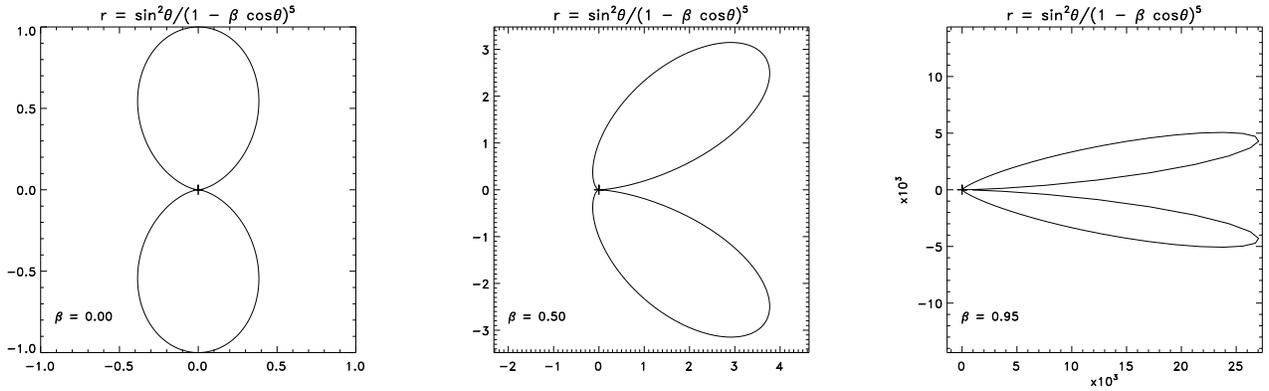
Figure 4.7: Polar plots of the radiation intensity as a function of direction, with acceleration parallel to $\mathbf{v}$, for different values of $\beta = v/c$. Velocity is in the x-direction.

maximum radiation is in the direction $\theta_m \simeq 1/(2\gamma)$ when $\beta \sim 1$. Here γ is the relativistic factor $(1 - v^2/c^2)^{-1/2}$. Moreover the intensity in this direction becomes extremely large as β gets close to one.

4.5.2 Acceleration Perpendicular to v

An even more important case is when $\mathbf{v}$ and $\dot{\mathbf{v}}$ are perpendicular. The geometry is illustrated in Fig. 4.8. One can then obtain

$$\frac{dP'}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0} \frac{\dot{v}^2}{4\pi c^3} \frac{1}{(1-\beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi\,(1-\beta^2)}{(1-\beta\cos\theta)^2}\right] \tag{4.83}$$
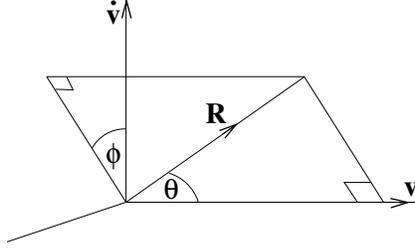
Figure 4.8: Definition of the angles for radiation when $\dot{\mathbf{v}}$ is perpendicular to $\mathbf{v}$.

where θ is the angle of $\hat{\mathbf{R}}$ with respect to $\mathbf{v}$ and ϕ is the polar angle of $\hat{\mathbf{R}}$ about $\mathbf{v}$ measured with respect to $\dot{\mathbf{v}}$ as zero.
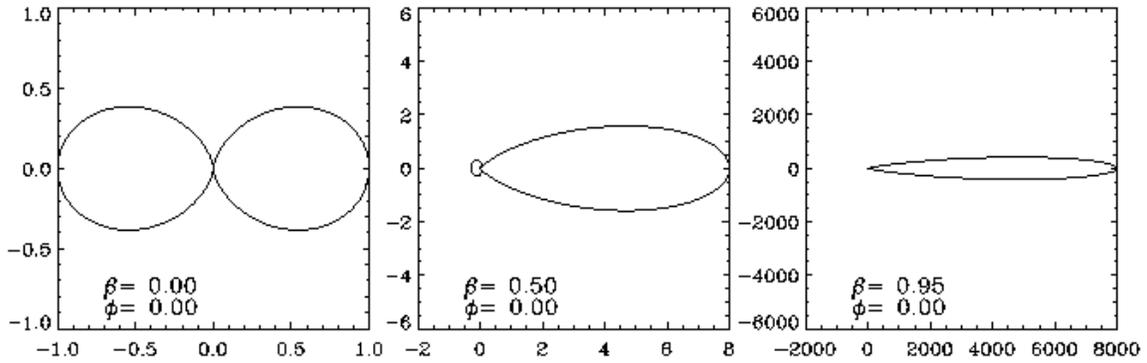
Figure 4.9: Polar plots of the radiation intensity as a function of direction, with acceleration perpendicular to $\mathbf{v}$, for the case where $\hat{\mathbf{R}}$ lies in the plane of $\mathbf{v}$ and $\dot{\mathbf{v}}$.

This distribution is likewise highly peaked in the forward direction for $\beta \sim 1$, having a typical half-angle extent of approximately $\frac{1}{\gamma}$.

4.5.3 Total Radiated Power

The general expression (4.76) can be integrated over solid angles by elementary but tedious methods to obtain

$$
\begin{aligned}
P' &= \frac{q^2}{4\pi\epsilon_o}\frac{2}{3c}\gamma^6\left[\left(\frac{\dot{v}}{c}\right)^2 - \left(\frac{\dot{\mathbf{v}}}{c}\wedge\frac{\mathbf{v}}{c}\right)^2\right]\\
&= \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\gamma^4\left[\left(\frac{\dot{v}}{c}\right)^2 - \left(\frac{\dot{\mathbf{v}}}{c}\cdot\frac{\mathbf{v}}{c}\right)^2\right] \quad,
\end{aligned}
\tag{4.84}
$$

the first form of which was obtained by Lienard (1898). These two alternate forms are convenient for obtaining the power when $\dot{\mathbf{v}}$ is parallel to $\mathbf{v}$:

$$
P' = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\frac{\dot{v}^2}{c^2}\gamma^6
\tag{4.85}
$$

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and $\dot{\mathbf{v}}$ is perpendicular to $\mathbf{v}$:

$$P' = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \frac{\dot{v}^2}{c^2} \gamma^4 \quad . \tag{4.86}$$

These expressions give important quantitative information about the rate of energy loss by a charge undergoing acceleration. The first thing we can see is that a charge could never be accelerated through the velocity c, because $\gamma \to \infty$ at $\beta \to 1$ and so infinite amounts of radiation would be emitted. This remark is quite independent of Einstein's theory of relativity which shows that the mass becomes infinite as $\beta \to 1$. Thus in 1898 when Lienard obtained his expression he already could have deduced that a charge could not be accelerated past $v = c$. Second, let us compare the rate of radiative energy loss to the energy gain from an accelerating electrostatic force.
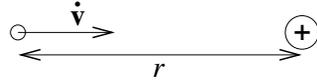
Figure 4.10: Comparing the radiative energy loss to the energy gained from force during acceleration due to a nearby charge.

Write the field as equivalent to the field a distance r from a charge Zq so that the acceleration is

$$\dot{v} = \frac{Zq^2}{4\pi\epsilon_0} \frac{1}{r^2 m_0 \gamma} \quad , \tag{4.87}$$

accounting for the relativistic mass increase. Supposing $\dot{\mathbf{v}}$ to be parallel $\mathbf{v}$, the rate of radiative loss is

$$P' = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c^3} \frac{Z^2 q^4}{(4\pi\epsilon_0)^2} \frac{1}{r^4} \frac{\gamma^4}{m_0^2} \quad . \tag{4.88}$$

This will equal the rate of gain of energy due to acceleration, namely

$$\frac{Zq^2}{4\pi\epsilon_0} \frac{v}{r^2} \quad ,$$

when

$$\left(\frac{Zq^2}{4\pi\epsilon_0 r} \right)^2 = Z \frac{3\beta \left(m_0 c^2 \right)^2}{2\gamma^4} \quad , \tag{4.89}$$

or

$$\frac{Zq^2}{4\pi\epsilon_0 r} = \left(\frac{3Z\beta}{\gamma^4} \right)^{1/2} m_0 c^2 \quad . \tag{4.90}$$

The left-hand side, here, is the potential energy of the charge and the right-hand side is a square-root factor times the rest-mass of the charge (expressed as an energy). For modestly relativistic particles, when we can take the square-root factor to be of order unity, we therefore see that radiation would begin to have an important effect relative to the parallel acceleration only when an electron (for example) is in a potential well at a depth $\sim m_0 c^2$ = 511 keV. Remembering that the binding energy of a hydrogen atom is only 13.6 eV this

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could happen only in the most exotic of situations (e.g. inner shells of heavy elements). Of course those situations would really have to be treated by quantum mechanics. Moreover these immensely strong electric fields ($\sim 10^{20}$ V/m) are never even approached in present accelerators. So radiation caused by acceleration parallel to v, such as in a linac, is never a serious consideration.

If $\mathbf{v}$ is perpendicular to $\dot{\mathbf{v}}$, however, the lowest order energy gain by the acceleration is zero. Compared with this the radiation may well be important. In the atomic force-field

$$P' = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c^3} \frac{Z^2 q^4}{(4\pi\epsilon_0)^2} \frac{1}{r^4} \frac{\gamma^2}{m_0^2} \tag{4.91}$$

and the classical kinetic energy in a circular orbit at radius r is

$$\mathcal{E} = \frac{1}{2} \frac{Zq^2}{4\pi\epsilon_0 r} \quad . \tag{4.92}$$
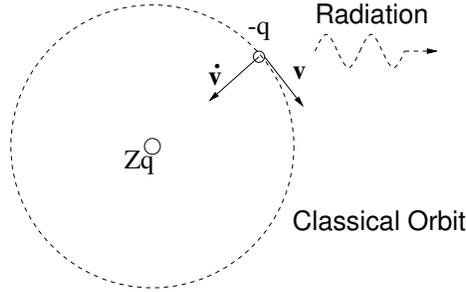
Figure 4.11: Radiation from a particle moving in a circular orbit arises from its perpendicular acceleration.

Hence the orbital energy is radiated with a characteristic time constant

$$
\begin{aligned}
\tau \sim \frac{\mathcal{E}}{P'} &= \frac{1}{2} \frac{Zq^2}{4\pi\epsilon_0 r} \left[\frac{q^2}{4\pi\epsilon_0} \frac{2c}{3} \left(\frac{Zq^2}{4\pi\epsilon_0 r} \right)^2 \frac{1}{r^2} \frac{\gamma^2}{m_0^2 c^4} \right]^{-1} \\
&= \frac{(m_0 c^2)^2}{2\gamma^2} \frac{3r}{2c} Z \left(\frac{Zq^2}{4\pi\epsilon_0 r} \right)^{-2} = \frac{(m_0 c^2)^2}{I^2} Z \frac{r}{c} \quad ,
\end{aligned}
\tag{4.93}
$$

where I is the binding energy of the particle in this circular orbit. For a "classical" hydrogen atom circular orbit, $I = 13.6$eV, $Z=1$, and $r = a_0 = 5.29 \times 10^{-11}$ m (the Bohr radius) we get $\tau = 1.5 \times 10^{-9} s$. Thus the rate of loss of energy by an electron in a "classical" Bohr orbit is such that the electron would spiral into the nucleus in a few nanoseconds. This, of course, was one of the key problems with classical electrodynamics that physics faced in the early 1900s, which prompted the eventual discovery of quantum mechanics.

As an immediately practical matter, we can also ask how fast a particle radiates energy because of being accelerated by a magnetic field, in the circular orbit of a cyclotron, for

example. In this case, the acceleration is $\dot{v} = v^2/d$, where d is the orbit radius. The power radiated is then, from eq (4.86),

$$P' = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \frac{v^4}{c^2 r^2} \gamma^4 = \frac{q^2}{4\pi\epsilon_0} \frac{2c}{3} \frac{\beta^4 \gamma^4}{r^2} \quad . \tag{4.94}$$

For a relativistic particle ($\beta \simeq 1$) the power therefore increases proportional to the fourth power of the energy (γ^4), and the energy loss per orbit for electrons moving with radius of curvature r can be written numerically in the form

$$\delta\mathcal{E}/\text{MeV} = 8.8 \times 10^{-2} \frac{(\mathcal{E}/\text{GeV})^4}{(r/\text{meters})} \quad . \tag{4.95}$$

This amounts to a major limitation for electron storage rings and accelerators above a few GeV energy. Jackson (p 668) cites the Cornell electron synchrotron with $r = 100$ meters having a loss of 8.8 MeV per turn at 10 GeV. The MIT Bates accelerator storage ring is designed for up to 1 GeV energy. With a bend radius of 9.1 m the loss is 9.8 keV per turn which is compensated by an accelerating stage within the ring.

4.6 Scattering of Electromagnetic Radiation

4.6.1 Thomson Scattering

We have seen that a non-relativistic accelerated charge radiates according to (eq 4.77),

$$\frac{dP}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} \left(\frac{\dot{\mathbf{v}}}{c}\right)^2 \sin^2 \alpha \quad , \tag{4.96}$$

where α is the angle between the direction of radiation and the direction of the acceleration, $\dot{\mathbf{v}}$. If the acceleration arises from an electric field E_i, then

$$\dot{\mathbf{v}} = \frac{q}{m} \mathbf{E}_i \quad , \tag{4.97}$$

Therefore the power radiated per unit solid angle from a single electron can be written:

$$\frac{dP}{d\Omega_s} = \frac{e^2}{4\pi\epsilon_0} \frac{1}{4\pi c} \left(\frac{e}{m_e c} E_i\right)^2 \sin^2 \alpha = \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 c\epsilon_0 E_i^2 \sin^2 \alpha \quad . \tag{4.98}$$

The combination of parameters arising in the last form of this equation,

$$r_e \equiv \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right) \tag{4.99}$$

has the dimensions of length, and is called the classical electron radius.
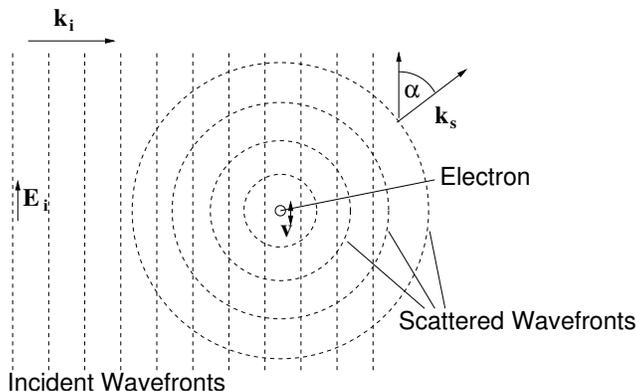
Figure 4.12: Schematic illustration of the process of Thomson Scattering.

A steady electric field will not give rise to radiation that is particularly interesting, but if the electric field is oscillating, it will give rise to radiation that is at a corresponding frequency.

The most elementary case one might consider is when the electric field varies sinusoidally with angular frequency ω. This is exactly the situation that arises if a charged particle such as an electron experiences the oscillating electric field of an incident electromagnetic wave at frequency ω. In this situation we speak of "scattering" of the incident wave by the electron. This process of acceleration of a free electron by an incident wave and reradiation of a wave into other directions is known as Thomson scattering.

Now the instantaneous power per unit area of the incident wave is given by the Poynting vector whose magnitude is

$$s_i = |\mathbf{E} \wedge \mathbf{B}/\mu_0| = \frac{1}{c\mu_0}E_i^2 = c\epsilon_0 E_i^2 \quad , \tag{4.100}$$

and we evaluate it at retarded time t' (i.e. at the time necessary to give rise to radiation at the field point at later time t). Therefore the scattered power per unit solid angle from a single electron can be written:

$$\frac{dP}{d\Omega_s} = r_e^2 \sin^2 \alpha \; s_i \quad . \tag{4.101}$$

The differential (energy) scattering cross-section is the ratio of $dP/d\Omega_s$ to the incident power density s_i. One can rapidly verify that this definition is in accord with the standard definition of a cross-section: that it should be such that the number of collisions per unit length is equal to the product of the cross-section and the density of targets. In this case the "projectiles" are represented by the incident energy of the wave. The projectiles can be considered to have a flux density proportional the wave power flux density, s_i. An alternative view of this cross-section is to regard it as the area across which the incident power flux density would have to flow in order to give rise to the power scattered. The cross-section is

$$\frac{d\sigma}{d\Omega_s} = r_e^2 \sin^2 \alpha \quad , \tag{4.102}$$

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where α is the angle between the scattering direction and the electric field (i.e. the polarization direction) of the incident wave.

Integrated over all scattering angles this expression yields the total Thomson scattering cross-section

$$\sigma = \frac{8\pi}{3}r_e^2 \quad . \tag{4.103}$$

If the electron is stationary apart from the oscillation that the wave imparts to it, then the scattered radiation will have exactly the same frequency (in this classical approximation) as the incident wave. However if the electron is moving prior to its perturbation by the incident wave, then there will be a Doppler shift of the scattered frequency both because the moving electron will experience the incident wave at a different frequency and because its radiation will be Doppler shifted at the observer. These two effects give a scattered frequency ω_s that is related to the incident frequency ω_i by

$$\omega_s = \omega_i + (\mathbf{k}_s - \mathbf{k}_i).\mathbf{v}_0 = \omega_i\frac{1 - \hat{\mathbf{k}}_i.\mathbf{v}_0/c}{1 - \hat{\mathbf{k}}_s.\mathbf{v}_0/c} \quad , \tag{4.104}$$

where $\mathbf{k}_i$ and $\mathbf{k}_s$ are the wave-vectors of the incident and scattered waves respectively, whose magnitudes are $k_i = \omega_i/c$ and $k_s = \omega_s/c$, and hats indicate unit vectors. The numerator and denominator of the fractional form for ω_s represent the two Doppler shifts just referred to. This one-to-one relationship between the scattered frequency and the component of the electron velocity along the direction $\mathbf{k}_s - \mathbf{k}_i$ is extremely helpful in plasma diagnostic applications. The velocity distribution of the electrons is directly revealed in the spectrum of Thomson scattered light.

4.6.2 Compton Scattering

One approximation implicit in our treatment of Thomson scattering is that all the incident wave does to the electron is to cause it to oscillate and that this oscillatory motion is added to an otherwise unperturbed prior motion. In other words, after the scattering has happened, the electron remains either stationary or moving at the same velocity as it had before. [In this section we will henceforward take the electron to be stationary prior to the scattering for simplicity.] But this cannot really be right, even on a classical picture, because we know that electromagnetic fields carry momentum. So if the wave is scattered, changing its momentum, then the electron's momentum must also be changed so as to conserve total momentum.

The classical effect can easily be calculated. By the symmetry of the $\sin^2\alpha$ angular distribution of scattering, the scattered radiation has zero momentum on average. Therefore the momentum imparted to the electron is just that of the incident radiation. We saw in section 3.2.3 that the momentum density of electromagnetic fields is equal to $1/c^2$ times the energy flux density. The force exerted by the incident radiation on the electron is equal to the total cross-section times the momentum flux density, which is c times the momentum density. So this force is

$$m_e\dot{v}_0 = \sigma s_i/c = \frac{8\pi}{3}r_e^2\epsilon_0 E_i^2\frac{1}{c} \quad . \tag{4.105}$$

In this classical picture there is a radiation pressure, applying over an area equal to the Thomson cross-section of the electron, which steadily pushes it in the direction of the incident radiation.

Quantum mechanics teaches us, however, that electromagnetic radiation is not smooth and infinitely divisible. Instead it takes the form of photons whose energy is $\hbar\omega$ when the angular frequency of the radiation is ω. If the size of the photon, the quantum of energy, is much less than the other energy scales in the problem, then the classical limit discussed above, can apply. If the photon energy is large, it cannot. Actually, the crucial question here is the *momentum* of the photon but this can be related to energy and compared with the rest energy of the electron ($m_0c^2 = 511$ keV) as we shall see. The quantum picture, then, is that each individual photon may, on encountering a free electron, bounce off in a scattering event. When it does so, the photon's momentum is changed, and the electron's momentum changes also so as satisfy conservation. As a consequence, for energetic (large momentum) photons, even an initially stationary electron recoils from a scattering event with substantial momentum. This recoil leads to a downshift in the energy (and hence frequency) of the scattered photon that will depend on the direction in which it is scattered.
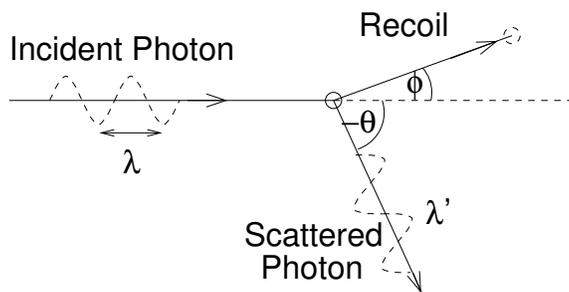
Figure 4.13: Compton scattering geometry in the scattering plane.

The kinematics of the problem, momentum and energy conservation, are all that is needed to relate the energy shift to the angle of scattering. Scattering takes place in a scattering plane. We will suppose that the photon is scattered through an angle θ and the electron recoils in a direction at an angle ϕ to the initial direction of the photon. We have to do the problem relativistically and we appeal to the general relativistic relationship relating energy and momentum:

$$\mathcal{E} = \sqrt{p^2c^2 + (m_0c^2)^2} \tag{4.106}$$

We denote the final momentum of the electron by p, the photon energy by $\mathcal{E}$ before and $\mathcal{E}'$ after the scattering collision. Then the momentum of the photon is $\mathcal{E}/c$ (from the energy relationship above or from our knowledge about the relationship between energy flux and momentum of electromagnetic fields). Then we write down the two components of momentum conservation parallel

$$\frac{\mathcal{E}}{c} = \frac{\mathcal{E}'}{c}\cos\theta + p\cos\phi \tag{4.107}$$

and perpendicular

$$0 = \frac{\mathcal{E}'}{c} \sin\theta + p\sin\phi \tag{4.108}$$

to the incident photon, and the energy conservation:

$$\mathcal{E} + m_0 c^2 = \mathcal{E}' + \sqrt{p^2 c^2 + (m_0 c^2)^2} \quad . \tag{4.109}$$

We eliminate ϕ by separating the ϕ terms in eqs 4.107 and 4.108 squaring and adding to get:

$$p^2 = \left(\frac{\mathcal{E}}{c}\right)^2 + \left(\frac{\mathcal{E}'}{c}\right)^2 - 2\frac{\mathcal{E}\mathcal{E}'}{c^2}\cos\theta \quad . \tag{4.110}$$

And then we eliminate the momentum p by squaring the square-root term of eq 4.109 to get

$$p^2 c^2 = 2m_0 c^2 (\mathcal{E} - \mathcal{E}') + (\mathcal{E} - \mathcal{E}')^2 \tag{4.111}$$

and subtracting from c^2 times the previous equation to get

$$0 = \mathcal{E}\mathcal{E}'(1 - \cos\theta) - m_0 c^2 (\mathcal{E} - \mathcal{E}') \quad . \tag{4.112}$$

This is the equation that relates the photon energy downshift to the angle of photon scattering. It is most often written in a form governing the photon wavelength $\lambda = 2\pi c/\omega = hc/\mathcal{E}$ and using $1 - \cos\theta = 2\sin^2\theta/2$,

$$\lambda' - \lambda = \frac{h}{m_0 c} 2\sin^2\frac{\theta}{2} \quad , \tag{4.113}$$

which expresses the "Compton Shift" of wavelength in terms of the "Compton Wavelength", $\lambda_c \equiv h/m_0 c = 2.426 \times 10^{-12}$ m, of the electron. A photon's wavelength equals the Compton wavelength when its energy is equal to the rest mass of the electron, $m_0 c^2 = 511$ keV. Therefore the Compton shift is important only for very energetic x-rays and for γ-rays.

The energy of the scattered photon is

$$\mathcal{E}' = \frac{m_0 c^2}{1 - \cos\theta + m_0 c^2/\mathcal{E}} \tag{4.114}$$

and the energy lost by the photon, and hence gained as kinetic energy by the electron is

$$\mathcal{E}\frac{1 - \cos\theta}{1 - \cos\theta + m_0 c^2/\mathcal{E}} \tag{4.115}$$

The cross section for this scattering must reduce to the Thomson cross-section at low photon energy. It was first calculated using relativistic quantum mechanics (1928) by Klein and Nishina shortly after Dirac's formulation of the relativistic quantum equations for the electron, predicting spin and negative energy states. The agreement of the Klein-Nishina cross-section with experiments was one of the early triumphs of Dirac's theory. For unpolarized radiation, the differential cross-section for photon scattering (which is different from
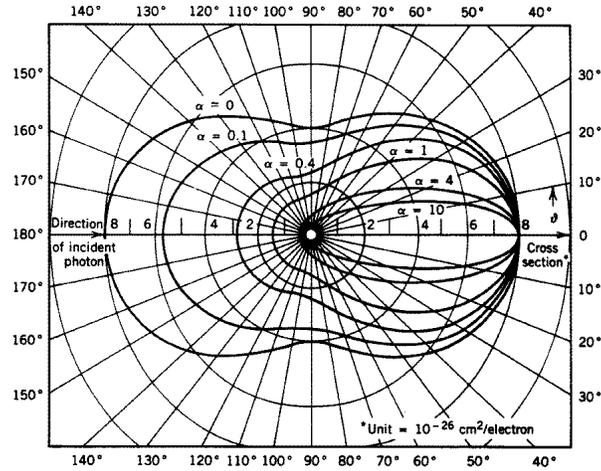
Figure 7.7 The Compton-scattering cross section for various incident energies. The polar plot shows the intensity of the scattered radiation as a function of the scattering angle θ. From R. D. Evans, *The Atomic Nucleus* (New York: McGraw-Hill, 1955).

Figure 4.14: Compton scattering cross-section angular variation. $[\alpha \equiv \mathcal{E}/m_e c^2]$.

the energy scattering cross-section by virtue of the photon energy shift) per unit solid angle is:

$$\frac{d\sigma}{d\Omega_s} = \frac{r_e^2}{2}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^2\left(\frac{\mathcal{E}}{\mathcal{E}'} + \frac{\mathcal{E}'}{\mathcal{E}} - \sin^2\theta\right) \quad . \tag{4.116}$$

In this form the reduction to the Thomson cross-section at low photon energy, so that $\mathcal{E}'/\mathcal{E} \to 1$, can be verified by integration of the Thomson formula over all possible incident radiation polarization directions. At high photon energy, $\mathcal{E} > m_e c^2$, forward or small angle scattering tends to dominate the cross-section, because the $(\mathcal{E}'/\mathcal{E})^2$ term becomes small at larger angles; although for those photons that are back-scattered $\theta \approx 180^o$, they lose practically all their energy to the electrons and retain only $\mathcal{E}' \to m_e c^2/2$. Figure 4.14 shows polar plots of the cross-section at different energies.

The Compton scattering process is a dominant attenuation mechanism in the 1 to 4 MeV photon energy range. It is sometimes helpful to distinguish between the cross-section for scattering of a photon, given above, and the cross-section for removal of energy from a photon beam, which is equal to the product of the scattering cross-section and the ratio of energy loss to initial photon energy. This later is sometimes called the Compton "absorption" cross-section since it represents the rate at which energy is transfered from photons to Compton scattered electrons. In either case, the attenuation of a photon stream of intensity I is governed by a differential equation:

$$\frac{dI}{d\ell} = -n_e\sigma I = -Zn_i\sigma I \quad , \tag{4.117}$$

where σ is the cross-section per electron, and the fact that the Z electrons are bound to each atom is ignored since the photon energy is so much higher than the electron binding

energy. The solutions to this equation are exponential ($\propto \exp(-n_e \sigma \ell)$) with inverse decay length $n_e \sigma$, which is called the "attenuation coefficient".
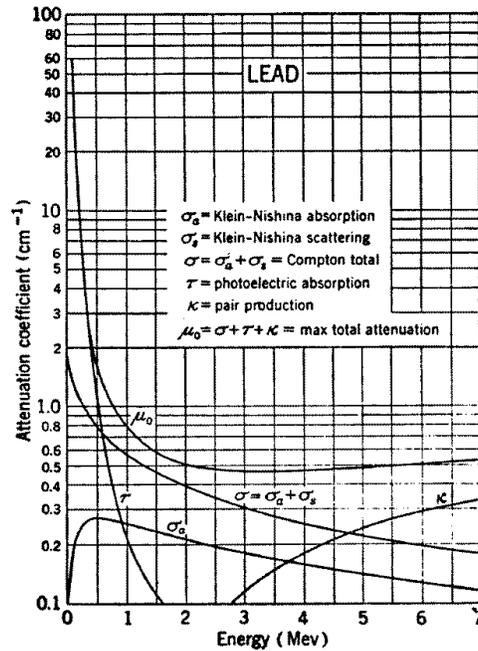
Figure 4.15: Photon attenuation coefficients for lead. [From Evans]

Since the ratio of mass to charge of most nuclei is very similar, between 2 and 2.8, the greatest attenuation arises from the greatest electron density, which corresponds to the greatest mass density. Hence lead, for example, has one of the largest attenuation coefficients. Figure 4.15 shows the total (angle-integrated) Compton attenuation coefficients together with the photoelectric absorption and pair production coefficients. These latter processes will be discussed later. The Compton attenuation, since it is simply the product $n_e \sigma$ can be scaled to any other material by multiplying by the ratio of electron densities, that is (for elements) by the quantity

$$\frac{A_{\text{lead}}}{\rho_{\text{lead}} Z_{\text{lead}}} \frac{\rho_{\text{other}} Z_{\text{other}}}{A_{\text{other}}} \quad , \tag{4.118}$$

where ρ is mass density, A is atomic weight, and Z is atomic number.

Chapter 5

Atomic Structure and Processes

5.1 Elementary atomic structure

Bohr Orbits correspond to principal quantum number n.

Hydrogen atom energy levels

$$\mathcal{E}_n = \frac{R_y}{n^2} \tag{5.1}$$

where the Rydberg energy is

$$R_y = \frac{m_e}{2} \left(\frac{e^2}{4\pi\epsilon_0 \hbar} \right)^2 = 13.6 \text{ eV} \quad . \tag{5.2}$$

The principal quantum number corresponds to the number of nodes of the radial solution of Schrödinger's equation. In the semi-classical treatment of Bohr, the energy of a circular orbit with angular momentum $n\hbar$ is equal to $\mathcal{E}_n$, but it is important to realize that in proper quantum mechanics, there are many possible different values of angular momentum for any principal quantum number n (from zero up to approximately $n\hbar$.)

The wave function for the hydrogen ground state can be written

$$\psi = \frac{1}{(\pi a_0^3)^{1/2}} \, e^{-r/a_0} \quad , \tag{5.3}$$

where a_0 is called the Bohr radius,

$$a_0 = \left(\frac{\hbar}{m_e} \right) \left(\frac{4\pi\epsilon_0}{e^2} \right) = 5.292 \times 10^{-11} \text{m} \quad . \tag{5.4}$$

Angular momentum of the electron orbit is quantized by a quantum number l. The actual angular momentum is $\sqrt{l(l+1)}$ times Planck's constant $\hbar$, which is approximately $l\hbar$ for large l. The quantum number l can take any integer value up to $n-1$. For the hydrogen atom the energy levels are (almost) unaffected by l but for multiple-electron atoms the energy differences between different l levels become large. In the wave-function, l is the
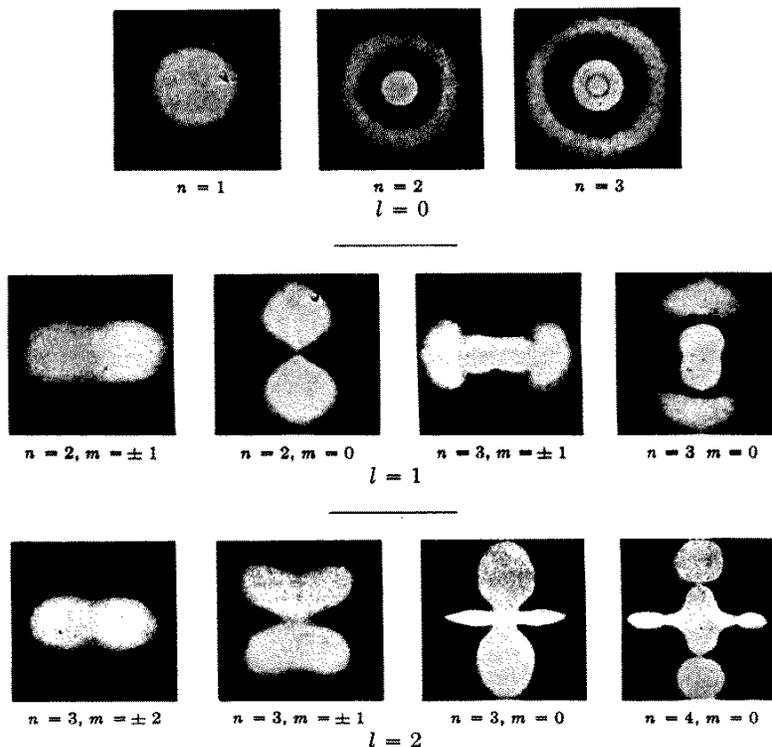
Figure 5.1: Representation of the electron cloud surrounding the nucleus for different quantum numbers. [From Herzberg]

number of the spherical harmonic $(P_l^m(\cos\theta))$ in its angular variation. Roughly speaking, l is the number of nodes of the wave-function.

A third quantum number is m which is roughly the component of angular momentum in a certain direction. It can take any value in the sequence $-l$, $-l+1$, ..., $l-1$, l. There are therefore $2l+1$ states for each l and $n+1$ possible l-values, totalling n^2 orbital states for each principal quantum level.

Electrons have spin giving rise to (non-orbital) angular momentum $\hbar/2$ and consequently 2 spin states $\pm 1/2$. They also satisfy Fermi-Dirac statistics, notably Pauli's exclusion principle, which is that no two electrons can occupy the same quantum state. The two spin states mean that up to two electrons can occupy any orbital state. However, the additional angular momentum of the spin greatly complicates the spectra by shifting the energies, giving rise to doublet states and the whole complexity of atoms.

5.2 Atomic processes in electromagnetic interactions

Atoms can emit or absorb radiation when their electrons make a transition from one quantum state to another. If the energies of the initial and final states are $\mathcal{E}_i$ and $\mathcal{E}_j$, then the energy of the photon emitted (or absorbed) is $h\nu_{ij} = |\mathcal{E}_i - \mathcal{E}_j|$ and obviously the conservation of

energy means that $\mathcal{E}_i$ must be greater than $\mathcal{E}_j$ for emission and less than it for absorption.

Absorption normally takes place by electric dipole transitions. If the energy density of radiation of frequency ν is $\rho(\nu)$ per unit frequency (ν), per unit volume, then the transition probability per unit time from the lower to the upper state is written $\rho(\nu_{ij})B_{ji}$ with the Einstein coefficient, B_{ji} given by Fermi's golden rule

$$B_{ji} = \frac{8\pi^3}{3h^2}\frac{S_{ij}}{4\pi\epsilon_0} = \frac{8\pi^3}{3h^2}\frac{1}{4\pi\epsilon_0}\left|\int \psi_i e\mathbf{r}\psi_j^* d^3r\right|^2 \quad . \tag{5.5}$$

The quantity S_{ij} is the square magnitude of the "matrix element" of the atomic dipole moment. (Single quantum states, are considered here to avoid worrying about statistical weight.)

Emission of radiation by the atom if the electron is in the upper state can also be induced by the presence of a radiation field. Its rate per unit time is $\rho(\nu_{ij})B_{ij}$ with the emission coefficient, B_{ij} equal to the absorption coefficient B_{ji}. Even in the absence of background radiation, "spontaneous" emission occurs with a probability per unit time of $A_{ij} = B_{ij}8\pi h\nu_{ij}/c^3$. A typical order of magnitude for $1/A_{ij}$, which is the lifetime of the excited state, is nanoseconds. As a result, atoms generally spend most of their time in the ground-state, the state of lowest energy, where electrons fill up all the low energy states as far as possible consistent with the exclusion principle.

Selection rules for which transitions are allowed by electric dipole radiation arise from the fact that the matrix elements, S_{ij}, are zero unless, for example, $\Delta l = \pm 1$. The energy levels are often illustrated graphically using what is called a "Grotrian" diagram, with energy (often measured in units of cm^{-1} corresponding to the inverse of the wavelength, $1/\lambda = \mathcal{E}/hc$) indicated by height, and angular momentum quantum level on the abscissa. Figure 5.2(a) shows the levels for hydrogen, which has equal energy for different l values. Figure 5.2(b) shows scaled energies for several different elements but with different levels of ionization so that they all possess three electrons, making them "lithium-like". The angular momentum degeneracy is broken, and the active electron can occupy only the principal level $n = 2$ and above, since the $n = 1$ level is already filled with the other two electrons.

When both the upper and lower energy states of the electron are bound states, the radiation emission occurs as a discrete narrow line at the specific frequency ν_{ij}. Line radiation is characteristic of a particular element and is one of the most powerful means of identification. However, transitions can also occur between bound states and free electron states. In the case of a downward transition, this is a process of "recombination" whereby an initially ionized atom recombines with a free electron, emitting the excess energy in the form of an electromagnetic photon, and forming a composite atom. In that case, because there is a continuous range of possible kinetic energies for the free electron (unlike the situation with a bound upper state) there is a continuous spectrum of electromagnetic radiation from this process. The opposite process is when a photon excites an initially bound electron into a free state. This is the process of photo-ionization of an atom, known more colloquially as the photoelectric effect.

For multiple-electron atoms the electrons are said to reside in "shells". These shells correspond to the principal quantum levels. The lowest energy states, corresponding to
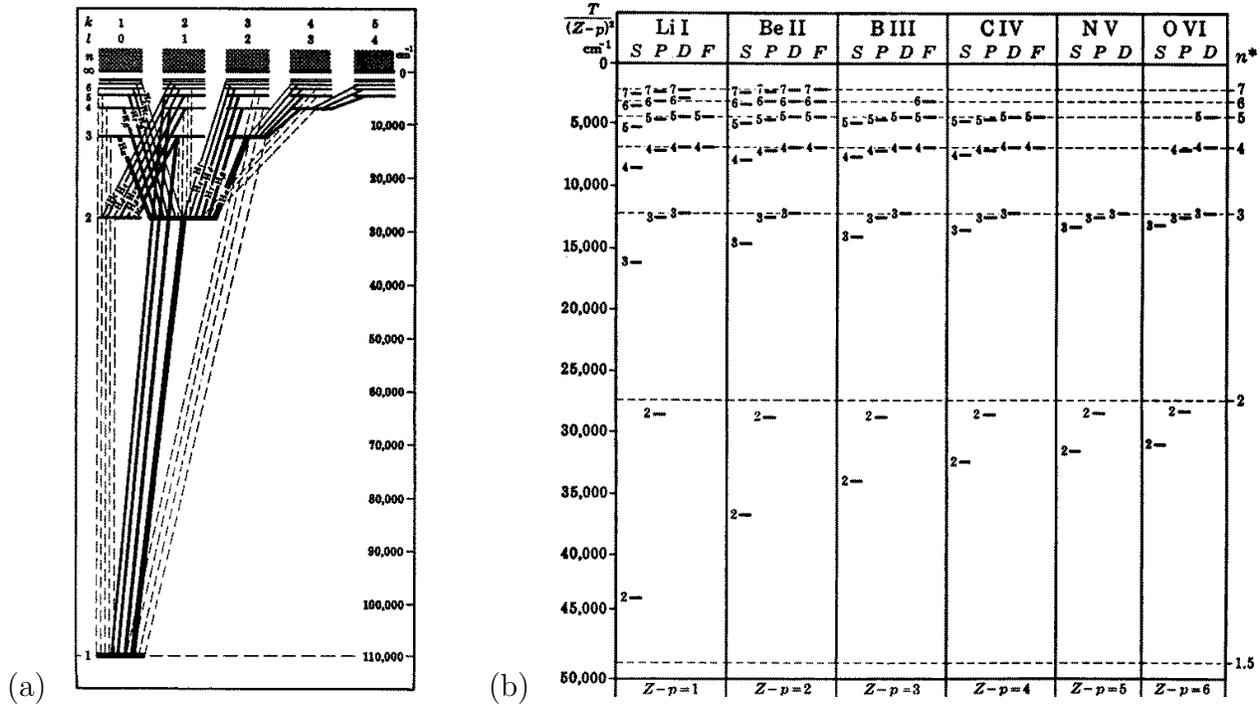
Figure 5.2: Energy level diagrams for (a) Hydrogen, indicating the allowed transitions, especially the Balmer series, (b) Lithium-like configurations arising from different stages of ionization, each possessing three electrons.[After Herzberg.]

electrons bound most strongly to the nucleus, are the $n = 1$ level, known as the K-shell. The next $n = 2$ level is known as the L-shell, and so on. An atom of a relatively heavy element has several shells filled with electrons. For example copper has the K, L, and M-shells filled and one electron in the N-shell. The easiest electron to remove from the atom is the electron in the highest energy state. Its binding energy determines the "ionization energy" of the element, that is, the energy that needs to be provided to strip the electron off (7.72 eV for copper). The K-shell electrons are much more strongly bound. Their binding energy is roughly $Z^2 R_y$, (more precisely, for copper 9.0 keV, corresponding to a wavelength of 0.138 nm).

If a photoelectric absorption removes an *inner* shell (e.g. K-shell) electron, as is often the case, then the resulting partially ionized atom is left in an excited state. Moreover, the excitation energy far exceeds the ionization energy of the resulting atom. There is, then, a hole in a very deep shell and electrons from other higher shells can liberate a lot of energy (roughly the K-shell binding energy) if they make a transition down into the hole. One of the easiest ways for this transition to occur is for the excess energy to be given to one of the weakly bound electrons in the highest shell. Since the energy exceeds the ionization energy, the weakly bound electron becomes completely unbound and is ejected from the atom with the excess energy less its binding energy appearing in its kinetic energy. This process is called the Auger effect (or sometimes "autoionization") and the ejected electrons are called

Auger electrons.

5.3   The Photoelectric Effect

It is interesting to contrast the photoelectric effect with Compton scattering. Both of these processes describe a photon interacting with an electron, transferring energy to it, and ejecting it from its atom. The key differences are

- In *Compton scattering* the combined momentum of the electron and photon is conserved, whereas in *photoelectric absorption*, momentum is transferred to the nucleus of the atom. Consequently:

- In *Compton scattering*, a photon emerges from the interaction carrying away substantial energy and momentum, whereas the *photoelectric effect* involves absorption of the entire photon energy and its transferral to binding and kinetic energy of the electron.

- *Compton scattering* is important only when the the photon energy is at least comparable to the electron rest energy, whereas the *photoelectric* cross-section increases strongly as the photon energy decreases, and completely dominates the photon absorption for energies less than roughly 100 keV.

The cross-section for photoelectric absorption is not straightforward to calculate rigorously. Some of the earliest calculations, prior to the full development of quantum mechanics (Kramers 1923), used classical radiation theory and the "correspondence principle" to obtain (quite accurate) estimates based on a calculation of the inverse process, radiative recombination, and hence deducing the photoelectric cross-section from equilibrium arguments and the principle of detailed balance. These non-relativistic calculations can be carried through using the bremsstrahlung formulism we shall discuss later, but the time is probably not well spent here. The result of these calculations is to obtain a cross-section for absorption by a single K-shell electron of photons with energy above the K-shell binding energy in the form

$$\sigma_p = \left[G \frac{32\pi^2}{3\sqrt{3}} \ r_e^2 \right] \alpha^3 Z^4 \left(\frac{m_e c^2}{h\nu} \right)^3 \quad . \tag{5.6}$$

where α is the fine structure constant ($\approx 1/137$), Z is the nuclear charge, $h\nu = \mathcal{E}$ is the photon energy, and G is a numerical factor of order unity. Notice that the square bracket factor is roughly seven times the Thomson cross-section ($8\pi r_e^2/3$). The most important feature, however, is the rapid increase of the cross-section ($\propto \mathcal{E}^{-3}$) as the photon energy decreases. This cross-section applies only down to a photon energy equal to the K-shell binding energy. There is therefore an "absorption edge" in the cross-section at that energy. Below that, the photon has insufficient energy for photoionization to occur for K-shell electrons and the absorption drops abruptly. However, L-shell electrons can be ejected down to an energy about $1/n^2 = 1/4$ of the K-shell energy, so they remain active, and N-shell electrons to even lower energies than that and so on, giving rise to an absorption edge for each shell. The higher
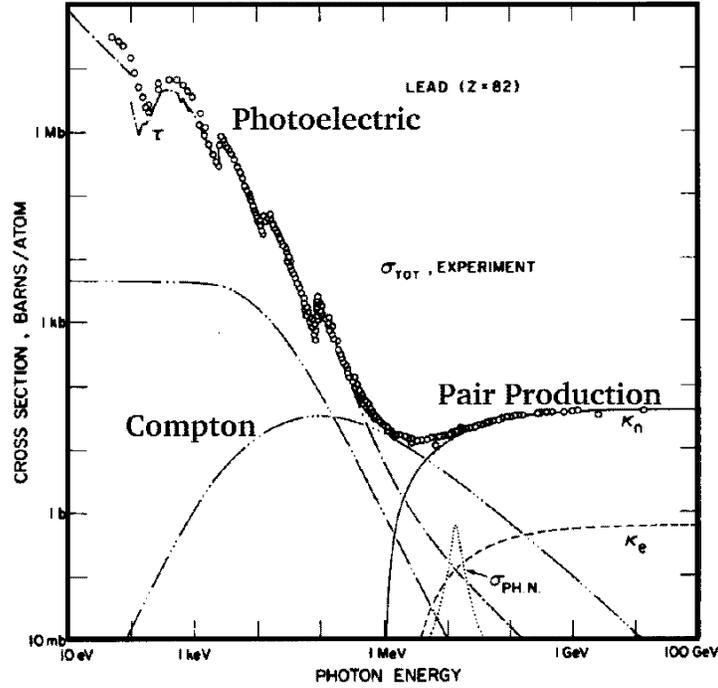
Figure 5.3: Photon absorption processes for Lead. At low energy the photoelectric effect dominates, and the absorption edges for the different shells are visible. [From H.Anderson, Ed.]

shells have intrinsically lower cross-section (by classically $1/n^3$ per electron) but because of the strong $\mathcal{E}^{-3}$ energy dependence, the total cross-section still maintains an upward trend at decreasing photon energy, as illustrated in Figure 5.3.

At higher energies, where relativistic effects are important for the ejected electrons, the dependence on photon energy becomes weaker. The reason for this is predominantly the fact that the electron's velocity becomes constant (equal to c) and, in the limit $\mathcal{E} \gg m_e c^2$, its momentum, which determines its quantum phase-space, becomes proportional to energy (rather than square-root of energy non-relativistically). The effect is to reduce the power of the photon energy dependence to $\sigma \propto \mathcal{E}^{-1}$.

A different, often cited, approximation for the non-relativistic photoelectric cross-section accounting for both K-shell electrons is [H.Hall, Rev. Mod. Phys. **8**, 358 (1936)]

$$\sigma_p = 4\sqrt{2}Z^5\alpha^4\left(\frac{m_e c^2}{h\nu}\right)^{3.5}\sigma_{\text{Thomson}} \quad . \tag{5.7}$$

while for the strongly relativistic case, $\mathcal{E} \gg m_e c^2$ the formula

$$\sigma_p = 1.5Z^5\alpha^4\left(\frac{m_e c^2}{h\nu}\right)\sigma_{\text{Thomson}} \tag{5.8}$$

100

may be used. [Each of these expressions is the cross-section *per atom*].

A comprehensive graphical representation for total photon absorption down to 1.8 keV energy is given in Figure 5.4
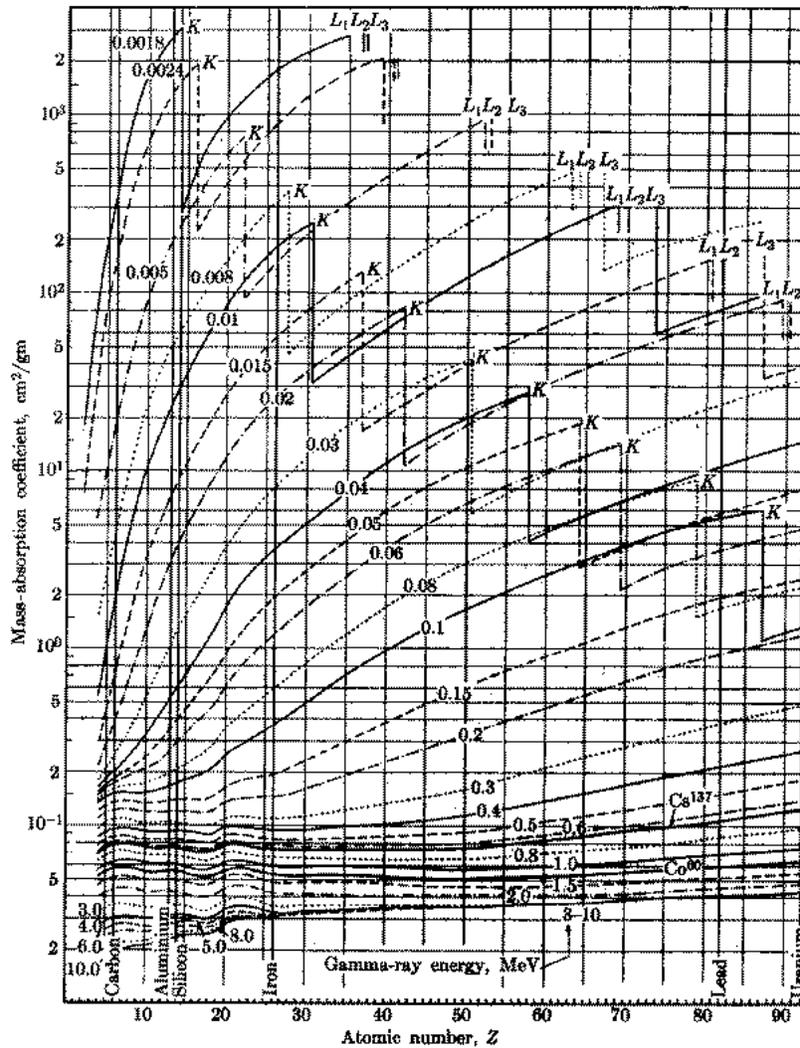
Fig. 7–8. Total mass-absorption coefficients for gamma rays in all elements from Be to U. Energy range 1.8 keV to 10 MeV. [From *Nucleonics* 19, 62 (1961). (Reproduced by permission of the McGraw-Hill Publishing Company, Inc.)]

Figure 5.4: Absorption coefficients for all elements. [From Enge]

5.4 Electrons and Pair Production

In 1928 Paul Dirac, by a master-stroke of mathematical insight, developed a theory of the electron that combines quantum mechanics consistently with relativity. The theory predicted many hitherto unsuspected phenomena including especially electron spin, but also that there

exist electron states of *negative* energy. In order to explain why electrons do not immediately make a transition from their positive energy state to a negative energy state, Dirac postulated that all the negative states are full. In that case, a vacancy or "hole" in the negative energy states, which naturally has *positive* energy with respect to a fully-occupied situation, behaves just like an electron except that its charge is *positive*. Thus the positron was predicted. Actually, Dirac first thought that the holes might correspond to protons. However a few years later, when the first experimental evidence of positrons was observed, he immediately saw this as confirmation of his theory.

When a positron — a hole in the otherwise full negative energy states — and an electron encounter one another, a downward transition of the electron into the lower, negative energy, state can occur. The electron fills up the hole, annihilating both particles; its energy becomes negative; and the rest energy $(2m_ec^2)$ and kinetic energy of the two particles appears as a photon. The analogy with a electron transition in an atom is helpful, although this situation really involves two free states of the electron, so it is more closely allied to bremsstrahlung.
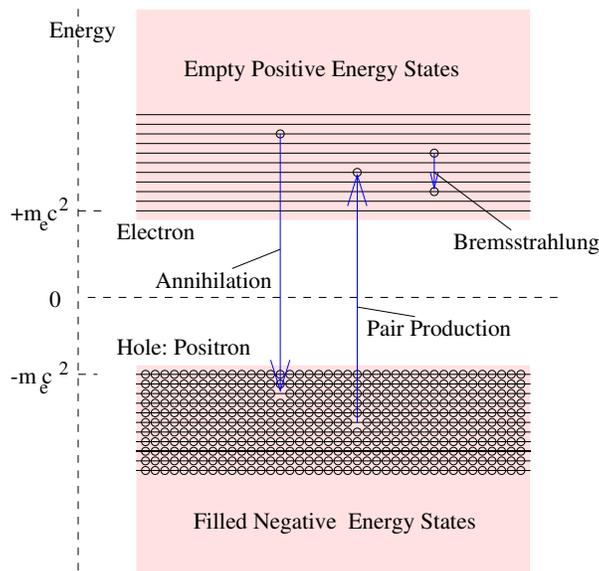
Figure 5.5: Schematic diagram of positive and negative energy electron states, annihilation and pair production.

Like any elementary quantum process, electron-positron annihilation has an inverse process: pair production. Pair production occurs when a photon with energy exceeding $2m_ec^2$ produces an electron and a positron, absorbing all the photon's energy, through interaction with a neighboring charge (usually a nucleus). The presence of the neighboring charge is necessary as a perturbation to couple the photon to the electron field and to absorb some of the photon's momentum, which cannot be completely transfered to the electron/positron pair because their ratio of momentum to energy is always lower than that of a photon $(1/c)$.

We are not in a position without extensive relativistic quantum mechanics to calculate
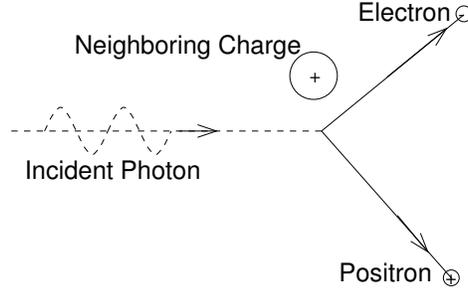
Figure 5.6: Pair production by an energetic photon in the presence of a neighboring charge.

the cross-section. Its value proves to be a weak (logarithmic) function of the photon energy in the range $2m_ec^2 \ll h\nu \ll m_ec^2 Z^{-1/3}\alpha^{-1}$

$$\sigma_{pp} \approx Z^2\alpha \; r_e^2 \left(\frac{28}{9} \ln \frac{2h\nu}{m_ec^2} - \frac{218}{27} \right) \quad . \tag{5.9}$$

Qualitatively, one can understand these factors as follows. The Z^2 factor arises from the strength of the perturbative coupling by a nucleus of atomic charge Z. The fine structure constant $\alpha \equiv e^2/4\pi\epsilon_0\hbar c$ and the classical electron radius $r_e \equiv e^2/4\pi\epsilon_0 m_ec^2$ are characteristic of coupling between electromagnetic photons and the electron. The cross-section becomes essentially constant for very high photon energy $h\nu \gg m_ec^2 Z^{-1/3}\alpha^{-1}$ because of shielding of the nucleus by its bound electrons. Naturally the cross-section falls to zero as the photon energy is lowered towards $2m_ec^2 = 1.02$ MeV. These characteristics are illustrated in Figure 5.3. That figure also shows the coefficient for pair production in interaction with electrons. It is negligible for heavy elements since the nucleus is slightly more than Z^2 times as effective as an electron and there are only Z electrons per nucleus. So the total attenuation due to all electrons is approximately $1/Z$ times that of the nuclei.

Table 5.1: Atomic Parameters: Definitions and Values

Rydberg Energy	R_y	$(m_e/2)\left(e^2/4\pi\epsilon_0\hbar\right)^2$	13.61	eV
Bohr Radius	a_0	$\hbar^2 4\pi\epsilon_0/e^2 m_e$	5.292×10^{-11}	m
Fine Structure Constant	α	$e^2/4\pi\epsilon_0\hbar c$	1/137.04	
Classical Electron Radius	r_e	$e^2/4\pi\epsilon_0 m_ec^2$	2.818×10^{-15}	m
Thomson Cross-Section	σ_T	$8\pi r_e^2/3$	6.652×10^{-29}	m
Relationships		$\alpha^2 = 2R_y/m_ec^2$		
		$2R_y a_0 = e^2/4\pi\epsilon_0 = \alpha\hbar c$		
		$r_e = a_0\alpha^2$		
		$\hbar = m_e c a_0 \alpha = m_e c r_e/\alpha$		

Chapter 6

Collisions of Charged Particles

The interactions of a moving charged particle with any surrounding matter are governed by the properties of collisions. We will usually call the incident particle the "projectile" and the components of the matter with which it is interacting the "target-particles" or just the "targets". The simplest situation one might imagine is that the matter consisted of free charged particles, electrons and nuclei. This is exactly the situation that applies if the matter with which the particle is interacting is a plasma. It might be thought that in this case, the mutual interaction of the target-particles themselves could be ignored, and the collisions treated as if they were all simple two-body collisions. This is not quite true because of the long-range nature of the electromagnetic force, as we shall see, but it is possible, nevertheless, to treat the collisions as two-body, but correct for the influence of the other target particles in this process.

In interactions with the atoms of solids, liquids or (neutral) gases, the fact that the target electrons are bound to the nuclei of their atom is obviously, in the end, important to the interaction processes. The atoms themselves can usually be treated ignoring the interactions between them, at least for projectiles with substantial kinetic energy. The simplest approximate analysis goes further, and starts from the highly simplified view that the electrons can be treated initially *ignoring* the force binding them to atoms. The corrections to this approach are naturally substantial, and the treatment cannot always yield accurate results. Nevertheless it represents a kind of baseline that more accurate calculations and measurements can be compared with.

The nuclei of the target are important in collisions with plasmas. However, in interactions with neutral atoms, direct electromagnetic interaction with the nucleus requires the projectile to penetrate the shielding of the orbiting electrons in the atom. Only particles with very high momentum can do that. Therefore the electrons of the target are usually the most important to consider, and tend to dominate the energy loss.

The topic of atomic collisions is an immense and complex one, in which quantum mechanics naturally plays a crucial role. It would take us far beyond the present intention to attempt a proper introduction to this topic. Two simplifying factors enable us, nevertheless, to develop this aspect of electromagnetic interactions in enough detail for many practical purposes. The first factor is that the details of atomic structure become far less influential

in collisions at energies much higher than the binding energies of atoms (which is about ten electron volts or so). The second is that even when quantum effects *are* important in the collisions, approximate formulas with wide applicability, but ignoring the details of particular atomic species, can be obtained by semi-classical arguments. The quantum corrections are then applied in a way that seems somewhat *ad hoc*, but often represents the way the earliest calculations were done, and gives simple analytic formulas.

6.1 Elastic Collisions

6.1.1 Reference Frames and Collision Angles

Consider an idealized non-relativistic collision of two interacting particles, subscripts 1 and 2, with positions $\mathbf{r}_{1,2}$ and velocities $\mathbf{v}_{1,2}$, which are not acted on by any forces other than their mutual interactions and which experience no changes in internal energy, so the collision is elastic. Their total (combined) momentum, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$, is constant, so that their center-of-mass,

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \quad , \tag{6.1}$$

moves at a constant velocity, the center-of-mass velocity:

$$\mathbf{V} \equiv \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \quad . \tag{6.2}$$

It is helpful also to introduce the notation

$$m_r \equiv \frac{m_1 m_2}{(m_1 + m_2)} \tag{6.3}$$

for what is called the "reduced mass". In terms of this quantity and the relative position vector $\mathbf{r} \equiv \mathbf{r_1} - \mathbf{r_2}$, the positions of the particles can be written:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_r}{m_1}\mathbf{r} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_r}{m_2}\mathbf{r} \quad , \tag{6.4}$$

and their velocities:

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_r}{m_1}\mathbf{v} \qquad \mathbf{v}_2 = \mathbf{V} - \frac{m_r}{m_2}\mathbf{v} \quad , \tag{6.5}$$

where $\mathbf{v} \equiv \dot{\mathbf{r}}$ is the relative velocity.

Some of our calculations need to be done in the center-of-mass frame of reference, in which $\mathbf{R}$ is stationary. Others need to be done in the lab frame or other frames of reference, for example in which one or other of the particles is initially stationary. The angles of vectors in these frames are important. The directions of all position vectors and of all velocity *differences* are the same in all inertial frames. However the directions of *velocities* are not the same in different frames.

For example, consider a collision illustrated in Fig 6.1. Collisions can be considered in a single plane-of-scattering which is perpendicular to the angular momentum of the system,

itself a constant. The angle of scattering, which we denote χ is just the angle between the initial direction of the relative velocity $\mathbf{v}$ and its final direction, $\mathbf{v}'$. This angle is different in different reference frames. Call the angle in the center-of-mass frame χ_c. By conservation of energy, the final relative velocity $\mathbf{v}'$ has absolute magnitude equal to that of the initial relative velocity, v_0. So the final velocity can be written in component form, in the center of mass frame, as

$$\mathbf{v}' = v_0(\cos \chi_c, \sin \chi_c) \quad , \tag{6.6}$$

where we have chosen the initial relative velocity direction for the x-axis.
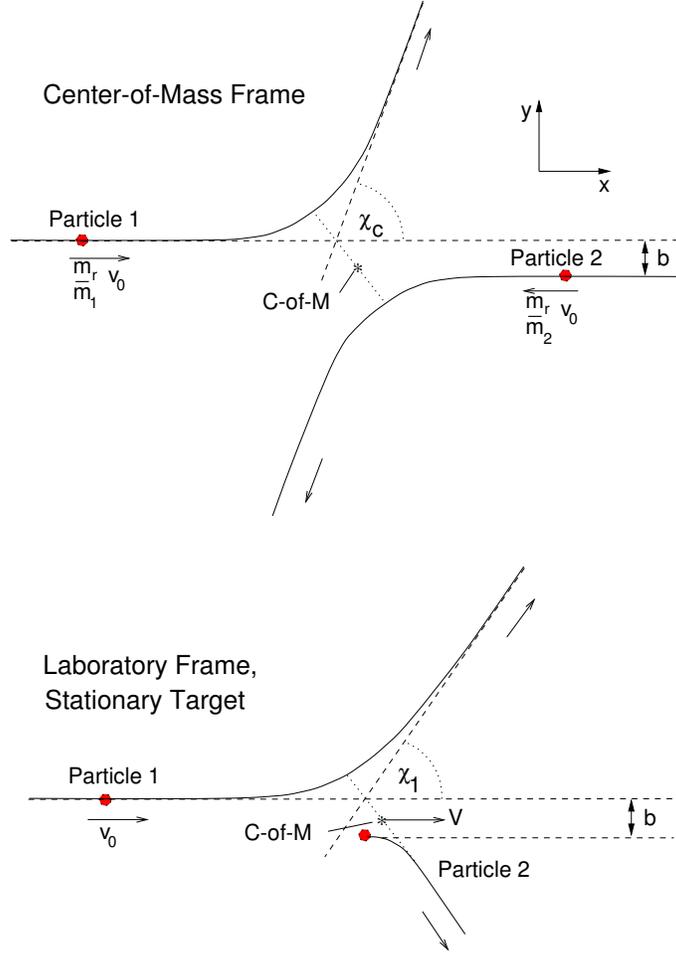
Figure 6.1: Collisions in center-of-mass and laboratory frames.

Substituting into eq(6.5) we find the final velocities in the lab frame to be given by

$$\mathbf{v}'_1 = \frac{m_r}{m_1}\mathbf{v}' + \mathbf{V} \;=\; \left(\frac{m_r}{m_1}v_0 \cos \chi_c + V, \frac{m_r}{m_1}v_0 \sin \chi_c\right) \quad , \tag{6.7}$$

$$\mathbf{v}'_2 = -\frac{m_r}{m_2}\mathbf{v}' + \mathbf{V} \;=\; \left(-\frac{m_r}{m_2}v_0 \cos \chi_c + V, -\frac{m_r}{m_2}v_0 \sin \chi_c\right) \quad . \tag{6.8}$$

The angle in the lab frame of the final velocity of particle 1 to its initial velocity (which is in the x-direction) , χ_1 say, is then just given by the ratio of the components of the final velocity,

$$\cot \chi_1 = \frac{\frac{m_r}{m_1} v_0 \cos \chi_c + V}{\frac{m_r}{m_1} v_0 \sin \chi_c} = \cot \chi_c + \frac{V}{v_0} \frac{m_1}{m_r} \csc \chi_c \quad . \tag{6.9}$$

For the specific case when particle 2 is a *stationary target*, with initial lab-frame velocity zero, the center-of-mass velocity is $V = m_1 v_0/(m_1 + m_2) = (m_r/m_2)v_0$ and so

$$\cot \chi_1 = \cot \chi_c + \frac{m_1}{m_2} \csc \chi_c \quad . \tag{6.10}$$

We often want to know how much energy or momentum is transferred from an incident projectile, (particle 1) to an initially stationary target (particle 2). Clearly from eq(6.8) we can obtain these quantities in terms of the scattering angle χ_c. So, the change in the x-momentum of particle 1 is simply

$$m_1 \left(\frac{m_r}{m_1} v_0 \cos \chi_c + V \right) - m v_0 = m_r v_0 \left(\cos \chi_c + \frac{m_1}{m_2} \right) \tag{6.11}$$

and the final recoil energy of particle 2 (which is the energy lost by particle 1) is

$$\begin{aligned}
Q \equiv \frac{1}{2} m_2 \left[\left(-\frac{m_r}{m_2} v_0 \cos \chi_c + V \right)^2 + \left(-\frac{m_r}{m_2} v_0 \sin \chi_c \right)^2 \right] &= \\
\frac{1}{2} m_2 \left(\frac{m_r}{m_2} v_0 \right)^2 \left[(-\cos \chi_c + 1)^2 + \sin^2 \chi_c \right] &= \\
\frac{1}{2} \frac{m_r^2}{m_2} v_0^2 2(1 - \cos \chi_c) &= \frac{1}{2} \frac{m_r^2}{m_2} v_0^2 4 \sin^2 (\frac{\chi_c}{2}) \quad . \tag{6.12}
\end{aligned}$$

Notice that the maximum possible energy transfer, which occurs when $\chi_c = 180°$, is

$$Q_{\max} = \frac{4 m_r^2}{m_1 m_2} \tfrac{1}{2} m_1 v_0^2 \quad . \tag{6.13}$$

All of these relations are completely independent of the nature of the interaction between the particles, since we have invoked only conservation of momentum and energy.

Impact Parameter and Cross-section

By definition, the cross-section, σ, for any specified collision process, when a particle is passing though a density n_2 of targets, is that quantity which makes the number of such collisions per unit path length equal to $n_2 \sigma$.[1] Sometimes a continuum of types of collision is under consideration. For example we can consider collisions giving rise to different scattering angles (χ) to be distinct. In that case, we speak in terms of **differential cross-sections**,

---

[1] An alternative definition can be invoked, equivalent to this first definition but in the frame of reference in which the single particle (1) is stationary and the particles of density n_2 are moving.

and define the differential cross-section $\frac{d\sigma}{d\chi}$ (for example) as being that quantity such that the number of collisions within an angle element $d\chi$ per unit path length is

$$n_2 \frac{d\sigma}{d\chi} d\chi \quad .$$

Sometimes other authors use different notation for the differential cross-section, for example $\sigma(\chi)$. However, our notation, with which we are familiar from calculus, is highly suggestive and the cross-sections obey natural rules for differentials implied by the notation.

For classical collisions, the **impact parameter**, b, shown in Fig 6.1, is a convenient parameter by which to characterize the collision. It is the distance of closest approach that would occur for the colliding particles if they just followed their initial straight-line trajectories. Alternatively, the impact parameter can be considered to be a measure of the angular momentum of the system in the center-of-mass frame, which is $m_r v_0 b$.
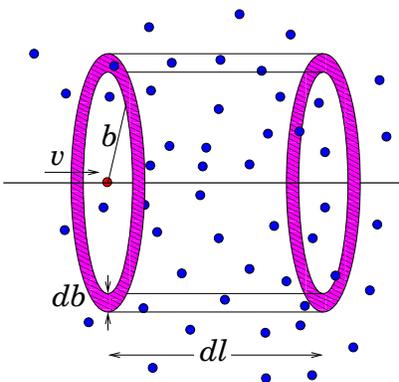
Figure 6.2: Differential volume for counting the number of collisions in length $d\ell$ with impact parameter b.

The differential cross-section with respect to the impact parameter is defined purely by geometry. As illustrated in Fig 6.2, one can think of the projectile (particle 1) as dragging along with itself an anulus of radius b and thickness db as it moves along a distance $d\ell$ of its path length. This anulus drags out a volume $d\ell 2\pi b db$, and the number of targets that are in this volume, and hence have been encountered in the impact parameter element db at b in this path-length is $n_2 d\ell 2\pi b db$. Consequently, from our definition, the differential cross-section for scattering at impact parameter b is

$$\frac{d\sigma}{db} = 2\pi b \quad . \tag{6.14}$$

Notice that the integral of this quantity over all impact parameters (i.e. $0 < b < \infty$) will certainly diverge, because it considers the projectile to be colliding with all the target particles it passes, no matter how far away they are. Therefore the total number of "collisions" of all possible types, per unit length in an infinite target medium is infinite. This mathematical singularity in the "total cross-section" points out the need to define more closely what

constitutes a collision, and alerts us to the fact that for collisions governed by interactions of infinite range, such as the forces between charged particles, we shall have to define our collisions in such a way as to account for some effective termination of the impact-parameter integration.[2] This termination, which is often expressed approximately as a cut off of the impact parameter integration at a maximum b_{max}, will be governed by consideration of the particle parameter whose change due to collisions we are trying to calculate. For example, the momentum or energy change in the collision may become negligible for $b > b_{max}$.

There is usually a one-to-one relationship between the impact parameter and the angle of scattering and hence with the energy transfer, Q, given by eq(6.12). Consequently the differential cross-section with respect to energy transfer, scattering angle and impact parameter are all related thus:

$$\frac{d\sigma}{dQ} = \frac{d\sigma}{d\chi_c}\left|\frac{d\chi_c}{dQ}\right| = \frac{d\sigma}{db}\left|\frac{db}{d\chi_c}\right|\left|\frac{d\chi_c}{dQ}\right| \tag{6.15}$$

If we are concerned with a quantity such as the energy of the projectile, which is changing because of collisions, and the change in each collision is an amount $Q(b)$ that depends on the impact parameter, then the total rate of change per unit length due to all possible types of collisions is obtained as

$$n_2 \int Q d\sigma = n_2 \int Q\, 2\pi b db \tag{6.16}$$

6.1.2   Classical Coulomb Collisions

The exact relationship between the impact parameter, b, and the scattering angle is determined by the force field existing between the colliding particles. For electromagnetic interactions of charged particles, the fundamental force is the Coulomb interaction between the forces, an inverse square law. As Isaac Newton showed, the orbit of a particle moving under an inverse square law force is a conic section; that is, an ellipse for closed orbits or a hyperbola for the open orbits relevant to collisions.

Elementary analysis shows that the resulting scattering angle χ_c for a collision with impact parameter b is given by

$$\cot\left(\frac{\chi_c}{2}\right) = \frac{b}{b_{90}} \quad , \tag{6.17}$$

where, for particles of charge q_1 and q_2 and initial collision velocity v_0 the quantity b_{90} is given by

$$b_{90} \equiv \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{m_r v_0^2} \quad . \tag{6.18}$$

Clearly from eq(6.17), b_{90} is the impact parameter at which the scattering angle in the center of mass frame is 90°. Trignometric identities allow us to deduce immediately from eq(6.17) that

$$\sin^2(\chi_c/2) = \frac{1}{1 + (b/b_{90})^2} \quad \text{and} \quad \frac{db}{d\chi_c} = -\frac{b_{90}}{2}\csc^2(\chi_c/2) \quad . \tag{6.19}$$

---

[2]It also shows the fundamental incoherence of the notion of the total number of collisions per unit length and concepts that depend on it such as the average change in some parameter *per collision*, which some authors unfortunately employ.

So that the energy transfer in a collision (see eq(6.12)) is

$$Q = \frac{1}{2}\frac{m_r^2}{m_2}v_0^2 4\frac{1}{1+(b/b_{90})^2} \tag{6.20}$$

and the rate of transfer of energy per unit length for a particle of energy $K \equiv \frac{1}{2}m_1 v_0^2$ colliding with stationary targets is

$$-\frac{dK}{d\ell} = n_2\frac{m_1 v_0^2}{2}\frac{4m_r^2}{m_1 m_2}\int\frac{2\pi b db}{1+(b/b_{90})^2} = n_2 K\frac{4m_r^2}{m_1 m_2}\pi b_{90}^2 \ln[1+(b_{max}/b_{90})^2] \quad , \tag{6.21}$$

where the upper limit of the b-integration, b_{max}, which prevents the integral diverging, will be discussed in a moment. One way to think of this equation is to regard the quantity $\pi b_{90}^2\frac{4m_r^2}{m_1 m_2}\ln[1+(b_{max}/b_{90})^2]$ as an effective collision cross-section for total energy loss. When multiplied by the density n_2 of targets it gives the inverse scale-length for energy loss, $d\ln K/d\ell$.
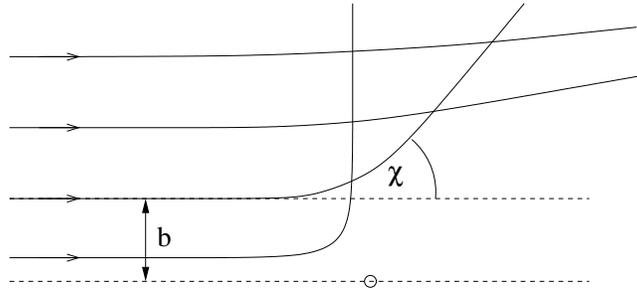
Figure 6.3: Scattering angle and impact parameter shown schematically for different Coulomb collisions.

The integral over impact parameters diverges if we extend it to infinite b. This is because the inverse square law has essentially infinite range. As a result, the dominant contribution to the energy loss cross-section comes from distant collisions, in which $b \gg b_{90}$, and hence the scattering angle is small. Several different physical effects can enter at large impact parameters to change the effective force-law and prevent the divergence. We will treat these effects separately in later sections, but in almost all cases, the exact value of the upper limit is not a very strong quantitative effect on the cross-section because b_{max}/b_{90} is large and appears inside a logarithmic term that may be written approximately $\ln(b_{max}/b_{90})$, which therefore varies very slowly with b_{max}. Many treatments adopt a small-angle approximation for the differential cross-section earlier in the derivation, leading to an expression $Q \propto 1/b^2$ and an integral that diverges both at small b and at large b. Such treatments then need to invoke a b_{min} cut-off of the integration, justifying it on the basis of a breakdown of the approximation, and naturally adopt b_{90} as that cut-off in this classical case. The resulting expression is then essentially identical to ours, which was obtained more rigorously. There are, in some circumstances, important physical effects that require us to cut-off the integration at small b *even before* b_{90} is reached. In those cases we simply replace the term $\ln[1+(b_{max}/b_{90})^2]$ with $2\ln(b_{max}/b_{min})$.

6.2   Inelastic Collisions

The effects that give rise to the cut-off of the Coulomb logarithm are primarily associated with the presence of other particles and forces in the system. If the target particles experience the force-field of another nearby particle, such as will be the case if the targets are electrons bound to the nuclei to form the atoms of a target material, then the dynamics of their binding gives rise to a cut-off. One way to think of this effect is to regard the electrons as behaving as if they were free only in collisions in which the energy transfer from the projectile is larger than their binding energy in the atom. Distant, small angle, collisions transfer less energy. A cut-off b_{max} should be applied at that impact parameter where the energy transfer is equal to approximately the binding energy.

Alternatively, and more physically, one can regard these collisions as being with a composite target system, the atom, in which there is a transfer of energy inelastically to the system, the energy being partially taken up in the ionization or excitation energy of the atom. Clearly, a fully rigorous calculation of such collisions requires the quantum structure of the atom to be considered, and so is intrinsically quantum-mechanical. Nevertheless, semi-classical calculations, taking quantum effects into account in a somewhat ad hoc manner, give substantial insight into the governing principles and, in fact, are able to give quantitatively correct forms for the cross-sections and energy loss.
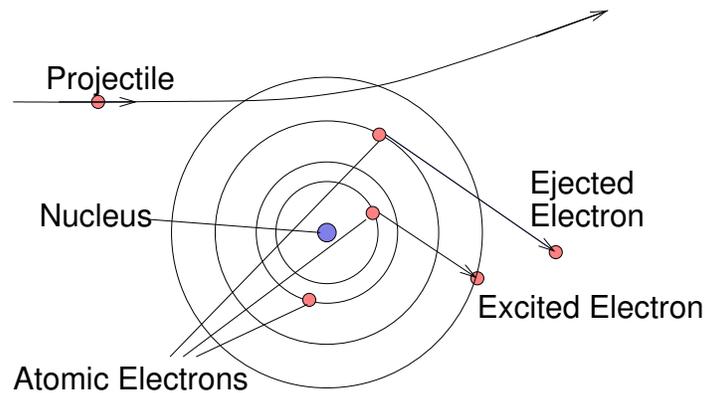
Figure 6.4: Collisions with an atomic system can excite or eject electrons from the atom.

6.2.1   Energy transfer to an oscillating particle

An approach to the problem of collisions with bound particles that can be treated classically, and becomes the basis for a quantum description, is to approximate the system as a charge bound in a simple harmonic potential well. Because we are mostly interested in large impact parameters, we regard the electric field of the projectile as uniform at the atom and then ask the question, in the encounter of the projectile with this oscillating electron, how much energy does the oscillator gain as a result of the fluctuating electric field of the passing projectile.

So consider a simple oscillating particle in a uniform electric field, $E(t)$. Its position x is governed by the equation

$$\ddot{x} + \omega^2 x = \frac{q}{m} E(t) \quad . \tag{6.22}$$

We solve this equation in the time range (t_1, t_2), with some assumed initial condition at t_1 so as to determine the energy gained by the particle at time t_2. This solution is readily obtained using what is called the "one-sided Green function" as follows. The solutions to the homogeneous problem (the equation with zero right hand side) are $\sin \omega t$ and $\cos \omega t$. The Green function is constructed as

$$H(t, \tau) = (\sin \omega t \cos \omega \tau - \cos \omega t \sin \omega \tau)/\omega \tag{6.23}$$

and the general solution is then

$$x(t) = A \sin \omega t + B \cos \omega t + \int_{t_1}^{t} H(t, \tau) \frac{q}{m} E(\tau) d\tau \quad , \tag{6.24}$$

where A and B are constants determined by the initial conditions. For simplicity consider the initial conditions to be that the oscillator is at rest: $A = B = 0$. Then at time t_2 the solution may be written

$$\frac{\omega m}{q} x(t_2) = \sin \omega t_2 \int_{t_1}^{t_2} \cos \omega \tau E(\tau) d\tau - \cos \omega t_2 \int_{t_1}^{t_2} \sin \omega \tau E(\tau) d\tau \quad . \tag{6.25}$$

When this expression is differentiated, the terms arising from the differentials of the limits cancel and we get

$$\frac{\omega m}{q} \dot{x}(t_2) = \omega \cos \omega t_2 \int_{t_1}^{t_2} \cos \omega \tau E(\tau) d\tau + \omega \sin \omega t_2 \int_{t_1}^{t_2} \sin \omega \tau E(\tau) d\tau \quad . \tag{6.26}$$

So the total (kinetic plus potential) energy in the oscillator can then rapidly be evaluated as

$$\begin{aligned}
\frac{1}{2} m(\omega^2 x^2 + \dot{x}^2) &= \frac{q^2}{2m} \left[\left(\int_{t_1}^{t_2} \cos \omega \tau E(\tau) d\tau \right)^2 + \left(\int_{t_1}^{t_2} \sin \omega \tau E(\tau) d\tau \right)^2 \right] \\
&= \frac{q^2}{2m} |E(\omega)|^2 \quad , \tag{6.27}
\end{aligned}$$

with the Fourier transform of the electric field written

$$E(\omega) = \int \exp(i\omega \tau) E(\tau) d\tau \quad . \tag{6.28}$$

We did this integration over a finite time, which avoids some mathematical difficulties, but we can now readily let $t_1 \to -\infty$ and $t_2 \to \infty$ and obtain the full domain Fourier integral. We have obtained the important general result that the energy transferred to a harmonic oscillator is proportional to the Fourier transform of the electric field evaluated at the resonant frequency of the oscillator, eq(6.27).

6.2.2 Straight-Line Collision
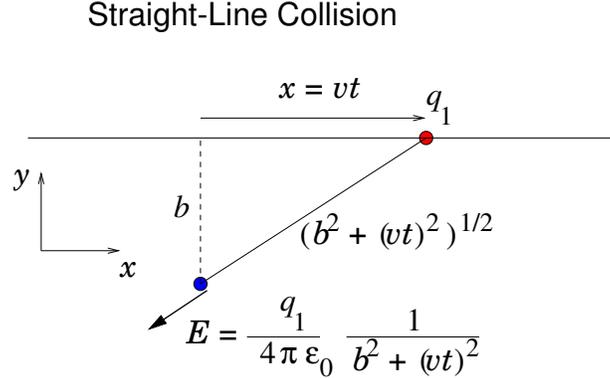
Figure 6.5: The approximation of a straight orbit gives a simple expression for the electric field as a function of time.

We are interested mostly in small-angle collisions, because, as we previously noted, they dominate the behavior, especially at the cut-off, $b_{\max}$. We approximate the orbit of the projectile in this case as a straight-line. Then, as illustrated in Fig 6.5, the electric field at the atom is just that due to a charge moving past at an impact parameter b and a constant speed. For a non-relativistic speed v the components of the electric field as a function of time are then

$$E_x(t) = \frac{-q_1}{4\pi\epsilon_0} \frac{vt}{(b^2 + v^2t^2)^{3/2}} \quad \text{and} \quad E_y(t) = \frac{-q_1}{4\pi\epsilon_0} \frac{b}{(b^2 + v^2t^2)^{3/2}} \quad . \tag{6.29}$$

the relativistic forms are qualitatively similar, and were calculated previously in section 4.2, see eq(4.39)

$$E_x(t) = \frac{-q_1}{4\pi\epsilon_0} \frac{\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \quad \text{and} \quad E_y(t) = \frac{-q_1}{4\pi\epsilon_0} \frac{\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \quad , \tag{6.30}$$

where γ is the relativistic factor $(1 - v^2/c^2)^{-1/2}$. The field components are plotted as a function of time in Fig 6.6. Clearly, by inspection of Fig 6.6, and eq(6.28) there will be a qualitative change in the behaviour of the Fourier transform of $E(t)$ and hence the energy transfer for $\omega b/\gamma v \gg 1$ compared with $\omega b/\gamma v \ll 1$. The characteristic time duration of the collision is $\sim b/\gamma v$. If this is much shorter than the characteristic oscillator time, $1/\omega$, we can take $\omega \approx 0$ and obtain by elementary integration

$$E_y(\omega) = \frac{-2q_1}{4\pi\epsilon_0 bv} \quad . \tag{6.31}$$

Because $E_x(t)$ is antisymmetric, $E_x(\omega) = 0$ in this small impact parameter limit. In the opposite limit, that is for collisions in which b is so large that $\omega b/\gamma v \gg 1$, $E(\omega)$ will be small because in eq(6.28) there are many oscillations of the factor $\exp(-i\omega t)$ within the

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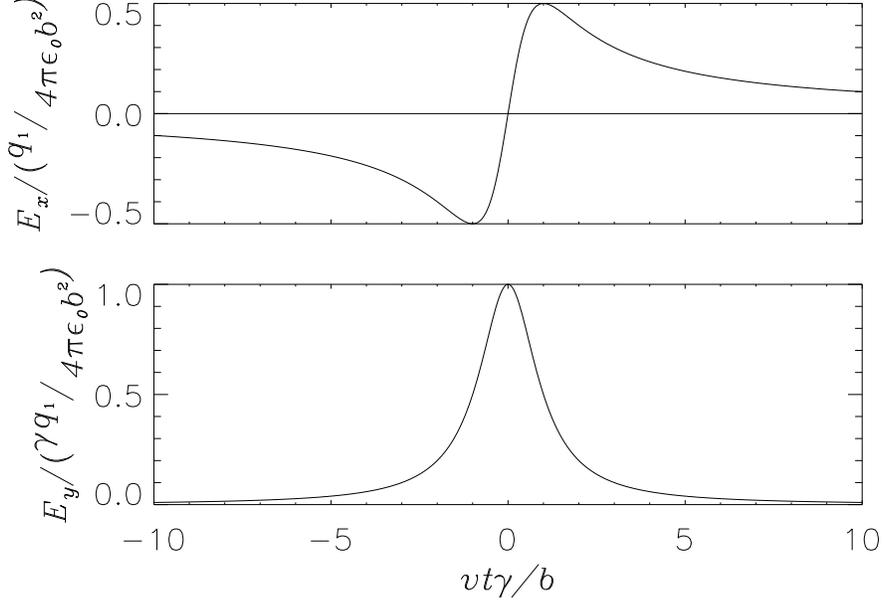
Figure 6.6: The electric field components in a straight-line collision.

smooth variation of $E(t)$. Thus we see that in collisions with a simple harmonic oscillator of frequency ω, there is a natural cut-off to the energy transfer at a maximum impact parameter

$$b_{\max} \approx \frac{\gamma v}{\omega} \quad . \tag{6.32}$$

Substituting eq(6.31) into eq(6.27), and restoring our notation of subscript 2 for the target and subscript 0 for the incident velocity, we obtain the energy transfer in a straight-line collision as

$$Q(b) = \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \frac{2}{m_2 v_0^2 b^2} = \frac{1}{2} \frac{m_r^2}{m_2} v_0^2 \, 4 \left(\frac{b_{90}}{b}\right)^2 \tag{6.33}$$

Notice that this is essentially the same expression as in eq(6.20) for the energy transfer to a free electron, except that the lower impact-parameter cut-off is not present here because of the assumption of a straight-line orbit for the projectile, which is unjustified at small impact parameters. The rate of energy loss is then obtained, as before, by integration over impact parameters from the minimum to the maximum corresponding to the limits of applicability of eq(6.31)

$$-\frac{dK}{d\ell} = n_2 K \pi b_{90}^2 \frac{m_r^2}{m_1 m_2} 8 \ln \left| \frac{b_{\max}}{b_{\min}} \right| = n_2 \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_2 v_0^2} \ln \left| \frac{\gamma v_0}{\omega b_{90}} \right| \quad . \tag{6.34}$$

6.2.3   Classical Energy Loss Rate Formula

One final consideration is needed before we have an energy loss formula useful for practical purposes. We have to have some way of applying the idealized harmonic oscillator calculation

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to actual atoms. An atom in general has a number Z, say, of electrons bound to the nucleus. Each electron may act as a target oscillator for energy transfer, and actually each electron may act as one of an infinite set of oscillators, corresponding to each of its possible quantum transitions. Energy transitions of magnitude $\mathcal{E}_i$ correspond to oscillators of frequency $\omega_i = \mathcal{E}_i/\hbar$, of course. To the ith transition may be assigned an *oscillator strength*, f_i, defined as the ratio of the actual rate of energy absorption by that transition to that of a corresponding harmonic oscillator. The semi-classical argument is then that each electron spends some fraction of its time behaving as if it were each of the possible oscillators, and consequently $\sum f_i = Z$. There is a more rigorous theorem in quantum physics called the (Thomas-Reiche-Kuhn) f-sum rule which states that the sum of all possible transition oscillator strengths from a specific level is equal to the number of electrons in the level. If this were applied blindly to all the electrons of the atom, it would give the same equation.

To obtain the total energy loss rate arising from collisions with a density of atoms n_a, whose atomic number is Z, we add up the contributions from all the possible transitions, weighted by the oscillator strength of that transition. Thus we obtain for the logarithmic term:

$$\sum_i f_i \ln \left| \frac{\gamma v_0}{\omega_i b_{90}} \right| = Z \ln \left| \frac{\gamma v_0}{b_{90}} \right| - \sum f_i \ln \omega_i = Z \ln \left| \frac{\gamma v_0}{\langle \omega \rangle b_{90}} \right| \quad , \tag{6.35}$$

where we have defined a kind of average oscillator frequency $\langle \omega \rangle$ by the equation

$$Z \ln \langle \omega \rangle \equiv \sum_i f_i \ln \omega_i \quad . \tag{6.36}$$

The total classical energy loss rate is then

$$\frac{dK}{d\ell} = n_a \left(\frac{q_1 e}{4\pi \epsilon_0} \right)^2 \frac{4\pi}{m_e v_0^2} Z \ln \Lambda \quad , \tag{6.37}$$

where we have substituted electron charge and mass for particle 2, and for brevity denoted the argument of the logarithm by

$$\Lambda = \frac{\gamma v_0}{\langle \omega \rangle b_{90}} \quad . \tag{6.38}$$

Actually, it turns out to be possible to evaluate the Fourier transforms of the relativistic fields in eq(6.30) in closed form and carry through the integration of the modified Bessel functions thus obtained [see Jackson]. When that is done, two very small corrections to our formula appear. The argument of the logarithm is multiplied by the factor 1.123 and an additional relativistic term is added, equivalent to the replacement

$$\ln \Lambda \to \ln \left| \frac{1.123 \gamma v_0}{\langle \omega \rangle b_{90}} \right| - \frac{v_0^2}{2c^2} \quad . \tag{6.39}$$

Neither of these corrections is quantitatively significant. The result was first obtained by Bohr in 1913, prior to the development of quantum mechanics. It is hardly complete as it stands, since the average $\langle \omega \rangle$ has to be estimated. However, because $\langle \omega \rangle$ appears only in the logarithm, even a rough estimate, for example setting $\hbar \langle \omega \rangle$ equal to the atom's ionization potential, will give a useful quantitative formula for the energy loss.

6.2.4 Quantum effects on close collisions

For the classical minimum impact parameter b_{90} to be applicable requires that the particles of the collision behave as point particles down to that impact parameter. However, quantum mechanics teaches us that particles do not behave like perfect points. The Heizenberg uncertainty principle states that the particle is localized only within a position uncertainty Δx if its momentum uncertainty is Δp such that $\Delta x \Delta p \approx \hbar$. Alternatively one can say that a particle with momentum $p = \gamma m v$ behaves like a wave with wave-vector $k = p/\hbar$. Or again, one can say that orbital angular momentum is quantized in indivisible units of $\hbar$. All of these are ways of indicating that in collisions the effective position of a particle is spread out over a distance of order $\hbar/p$. Consequently, quantum effects prevent us from extending the classical integration over impact parameters below a value of

$$ b_q \approx \frac{\hbar}{\gamma m_r v} \quad , \tag{6.40} $$

(using the reduced mass as the appropriate one for quantum cut-off momentum). The classical b_{90} lower impact parameter cut-off will be applicable only if

$$ \frac{b_{90}}{b_q} \approx \frac{q_1 q_2}{4\pi\epsilon_0 \hbar v} = \frac{q_1 q_2}{e^2} \alpha \frac{c}{v} > 1 \quad , \tag{6.41} $$

where α is the fine structure constant, approximately $1/137$, This criterion is a requirement that the collision velocity with electron targets should be less than $Z_1 c/137$.

In practice this means that electrons with energy greater than 13.6 eV, protons with energy greater than 25 keV, or alpha particles with energy greater than 400 keV will *not* be appropriately treated using the classical lower impact parameter cut off. Instead, an approximation to the quantum-mechanical result may be obtained by simply cutting off the impact parameter integration at b_q rather than b_{90}. If we choose[3] $b_q = \hbar/2\gamma m_e v$, then in collisions of heavy particles with atoms, for which $m_r = m_e$,

$$ \ln \left| \frac{b_{\max}}{b_{\min}} \right| = \ln \left| \frac{2\gamma^2 m_e v_0^2}{\hbar \langle \omega \rangle} \right| \quad . \tag{6.42} $$

This value is then consistent with that obtained for the relativistic case using a quantum scattering treatment and the first Born approximation, by Bethe (1930),

$$ -\frac{dK}{d\ell} = n_a \left(\frac{q_1 e}{4\pi\epsilon_0} \right)^2 \frac{4\pi}{m_e v_0^2} Z \left(\ln \left| \frac{2\gamma^2 m_e v_0^2}{\hbar \langle \omega \rangle} \right| - \frac{v_0^2}{c^2} \right) \quad , \tag{6.43} $$

where again the final term, v_0^2/c^2, which we have not derived, is at most a small correction.

If the projectile is an electron or positron, then the quantum cut-off must be estimated in the center-of-mass frame, and the expression becomes

$$ -\frac{dK}{d\ell} = n_a \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{4\pi}{m_e v_0^2} Z \left(\ln \left| \left(\frac{\gamma+1}{2} \right)^{1/2} \frac{(\gamma-1) m_e c^2}{\hbar \langle \omega \rangle} \right| - \frac{v_0^2}{2c^2} \right) \quad . \tag{6.44} $$

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[3]The factor of 2 here is our only real artifice. It gives the argument of the logarithm equal to that obtained by a full quantum calculation

6.2.5 Values of the Stopping Power

We have so far left open the question of what value to take for $\hbar\langle\omega\rangle$. Bloch (1933) showed from an analysis of the Thomas-Fermi model of the electron charge distribution in an atom that one would expect that $\hbar\langle\omega\rangle \propto Z$. In recognition of the work of Bethe and Bloch, eq 6.43 is often referred to as the Bethe-Bloch formula. The formula is often written as

$$-\frac{dK}{d\ell} = n_a \left(\frac{q_1 e}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} B \quad, \tag{6.45}$$

with the quantity B, called the "atomic stopping number", corresponding to the factor

$$Z\left(\ln\left|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right| - \frac{v_0^2}{c^2}\right) \quad.$$

Also B/Z is then called the "stopping power" per (atomic) electron, recognizing that an atom has Z electrons. The stopping power is determined from experiments, and the appropriate value to use for $\hbar\langle\omega\rangle$ is determined from those measurements.

A complication that we have not discussed arises because our treatment has assumed that the orbital velocity of the electrons in the atom can be ignored relative to the velocity of the incident particle. This is not the case when dealing with the inner shell electrons of high-Z atoms or very low incident-energy projectiles. Then a reduction in the stopping number occurs because (for example) the (innermost) K-shell electrons are ineffective in removing the projectile's energy. This effect is numerically compensated by substracting a correction term C_K so that

$$B = Z\left(\ln\left|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right| - \frac{v_0^2}{c^2}\right) - C_K \quad. \tag{6.46}$$

In this form, the value of $\hbar\langle\omega\rangle$ is empirically determined to be about $11.5 \times Z$ eV, and C_K is a function of the quantity $\xi \equiv (c^2/v_0^2)(Z - 0.3)^2\alpha^2$ (which represents the squared ratio of the K-shell velocity to the projectile velocity). A simple approximate form for C_K is

$$C_K(\xi) = \frac{2.3\xi}{1 + 1.3\xi^2} \quad, \tag{6.47}$$

correct to within 10% from $\xi = 0$ to $\xi = 2$. It tends to zero at high projectile energy and peaks at about unity at low velocity, where $\xi \approx 1$. These and many other details have been reviewed by Evans (1955).

6.2.6 Effects of surrounding particles on distant collisions

Let us return now to our primitive energy loss rate calculation, eq 6.34 which may be considered in the form

$$\frac{dK}{d\ell} = n_2 \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_2 v_0^2} \ln\left|\frac{b_{\max}}{b_{\min}}\right| \quad. \tag{6.48}$$

In the preceding sections we have been discussing appropriate choices of $b_{\min}$ based either on the classical effects of large scattering angles (giving b_{90}) or on quantum-mechanical effects of the de Broglie wave-length of the projectile/target combination. We also discussed the appropriate $b_{\max}$ based on the effects of the binding of electron targets to their nuclei. However, another effect can sometimes be more important than the atomic binding structure in determining $b_{\max}$, namely the influence of surrounding particles.

We have tacitly assumed so far that the interaction of the projectile and any specific target can be treated *ignoring* the effects of the other targets in the vicinity. We have calculated the projectile/target interaction in isolation and then presumed that we can add up the effects of all the different targets via a simple impact-parameter integration. This may not be the case. For example, it definitely is not the case when the electrons of the target are *unbound*; or in other words for a plasma target. In that case there is no intrinsic cut-off to the the collision integral arising from the oscillator effects introduced in section 6.2.1 and the effect of the nearby particles essentially always determines $b_{\max}$. Even in collisions with atomic matter, especially for relativistic electrons, the effect of nearby particles can significantly lower the energy transfer rate. In the atomic collision context the corrections are often referred to as the "density effect" because they are most significant for high-density matter.

It is still the case that transfer of energy to the target arises from the electric field produced by the incident projectile. However, what we need to do is to account for the influence of the other particles in the target medium on the electric field that the projectile produces at a specific target. Expressed in this way, it is immediately clear that what we need is to take account of the *dielectric properties* of the target medium. The individual particles of the medium respond to the influence of charge (the projectile in this case) so as to alter the electric field in the medium from what it would otherwise have been. This is exactly what we mean by the dielectric response of the medium.

Of course, though, it is not the steady-state dielectric response that we require but the response at the high frequencies of interest in the collisions. Moreover, when we think about a target medium consisting of a density of idealized oscillators, as we did before, it is the properties of those oscillators themselves that determines the dielectric response at frequencies close to their resonant frequencies. Thus the dielectric response and the energy-loss collisional response are not two *separate* properties of the medium; they are intimately connected.

The idealized oscillator model can be generalized to discuss a medium with any relative dielectric permittivity $\epsilon(\omega)$ having a resonant form ($\epsilon - 1 \propto (\omega - \omega_i)^{-1}$), and an expression for the rate of loss of energy of an incident projectile to this resonance can then be obtained. Fermi (1940) first gave the following formula, which would take us too long to rederive, for the energy loss attributable to collisions with impact parameter greater than a as

$$\left.\frac{dK}{d\ell}\right|_{b>a} = \frac{2}{\pi}\frac{q_1^2}{4\pi\epsilon_0 v_0^2}\Re\int_0^\infty is^* K_1(s^*)K_0(s)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)d\omega \quad, \qquad (6.49)$$

where $\Re$ denotes real part, $\beta = v_0/c$, K_1 and K_2 are modified Bessel functions, and their

argument is s such that

$$s^2 \equiv \frac{a^2\omega^2}{v_0^2}[1 - \beta^2\epsilon(\omega)] \quad . \tag{6.50}$$

It can be shown, but not trivially, [Jackson] that this $dK/d\ell$ reduces to the the Bohr expression (eq 6.39) if the $\beta^2\epsilon(\omega)$ term in s is neglected.

Rather than pursue the topic for the atomic case, let us consider a simple argument for a plasma. The dielectric constant for a (magnetic field-free) plasma at high frequency is

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad , \tag{6.51}$$

where

$$\omega_p \equiv \frac{n_e e^2}{m_e \epsilon_0}$$

is called the plasma frequency. Therefore when the field frequency of interest is less than ω_p the dielectric constant is negative and wave electric fields no longer propagate in the medium; instead they decay exponentially with distance from their source. In collisions, as we have seen before, the frequency of the interaction electric field is approximately v_0/b. Therefore, for impact parameter, b, greater than v_0/ω_p we would expect that the effectiveness of the collisions would fall off because of the dielectric effects. Applying this value for $b_{\max}$ we obtain an energy loss rate expression corresponding to eq 6.37 as

$$\frac{dK}{d\ell} = n_e \left(\frac{q_1 e}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} \ln\Lambda = \frac{q_1^2}{4\pi\epsilon_0} \frac{\omega_p^2}{v_0^2} \ln\Lambda \tag{6.52}$$

but with Λ given approximately by

$$\Lambda = \frac{v_0}{\omega_p b_{90}} \quad . \tag{6.53}$$

What we have done, in effect then, is to replace the value $b_{\max} = \gamma v_0/\langle\omega\rangle$ in the definition of Λ, eq (6.38) with

$$b_{\max} = \frac{v_0}{\omega_p} \quad . \tag{6.54}$$

The factor by which the logarithmic argument Λ of the Bethe-Bloch formula is multiplied is therefore $\gamma\omega_p/\langle\omega\rangle$. But the density effect can only lower the absorption rate so we should more properly have used $b_{\max} = \min(v_0/\omega_p, \ \gamma v_0/\langle\omega\rangle)$. The electrons behave as if they are free when $\omega > \omega_{ij} \sim \langle\omega\rangle$. Hence plasma-like, i.e. free-electron, behaviour occurs only when $\omega_p > \langle\omega\rangle$, which is when the plasma expression for $b_{\max}$ applies, because it is the smaller.

A rough estimate of the ratio of $\omega_p/\langle\omega\rangle$ may be obtained by taking the density of atoms in a solid to be about 10^{30} m^{-3}, and the electron density to be Z times that. Then

$$\hbar\omega_p \approx 37Z \quad \text{eV}. \tag{6.55}$$

For medium weight solid elements, $\hbar\langle\omega\rangle \sim 11Z$ eV so we expect the plasma effect to be slightly noticeable since on this basis $\omega_p/\langle\omega\rangle > 1$. The question is a little more complicated than this, though because not all the electrons are going to behave as if free so we have somewhat over estimated the density of the electrons that behave as if free. In extreme relativistic cases, $\gamma \gg 1$ the plasma (density) effect will always dominate.

6.3 Angular Scattering from Nuclei

Up to this point we have been discussing the energy loss of the projectile and have focussed on its interactions with electrons. This focus on electron targets is entirely appropriate for calculating energy loss because, as illustrated by eq (6.21) or (6.34) the rate of energy loss is, classically, inversely proportional to the mass of the target particle[4]. Therefore the loss of energy is in fact predominantly to the light particles, electrons, and this predominance depends only on the elementary dynamics of collisions. However, in addition to losing energy, the projectile also generally experiences angular scattering in the direction of its velocity. If this angular scattering is our concern, as it was in Rutherford's original experiments on the angular scattering of alpha particles which established that the nucleus is far smaller than the atom, then collisions with the heavy particles in our scattering medium, the nuclei of the atoms or the ions of a plasma, are important. This process, illustrated in Figure 6.7, is often called "elastic scattering", although the expression may be considered somewhat misleading in that some energy is lost by the projectile in the collision, and the process is no more elastic than a collision with a free electron, for instance.
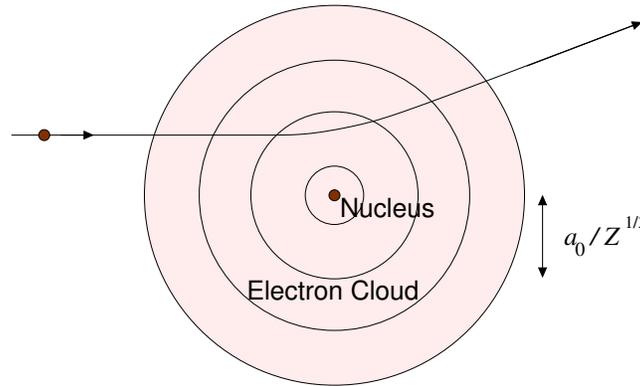
Figure 6.7: Angular scattering from nuclei occurs only if the impact parameter is less than the size of the electron cloud.

Qualitatively, the relative importance of energy loss and angular scattering can be grasped by imagining the difference between a tennis ball colliding with a random arrangement of cannon balls, or a cannon ball colliding with a random arrangement of tennis balls. In the first case, the light projectile will bounce around changing its direction of motion many times before losing its energy; while in the second case, the heavy projectile will plough through the light targets, losing energy faster than its direction is deflected.

The angular scattering of a particle in a classical coulomb collision is governed by the Rutherford formula for the differential scattering cross-section per unit solid angle at a

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[4]This proportionality can be traced to the inverse dependence of the energy transfer in a collision on m_2, but only because cancellation of reduced mass factors occurs in the product Qb_{90}^2.

scattering angle *in the center of mass frame*, χ_c,

$$\frac{d\sigma}{d\Omega} = \frac{b_{90}^2}{4\sin^4\chi_c/2} \quad .$$ (6.56)

This formula may readily be derived from the considerations in section 6.1.1. It shows that the predominant scattering is through small angles. Those small angles arise from large impact parameters. There are some collisions, of course, which arise from small impact parameters, close to b_{90}, that give rise to large scattering angles, but these are far fewer in number than the small-angle collisions; so by the time the probability of scattering by a large angle is significant, multiple scatterings by small angles will have caused a kind of diffusion of the direction of the particles in perpendicular velocity. Figure 6.8 illustrates an idealized situation, in which the projectile loss of energy is taken as zero, so its velocity vector has constant magnitude and moves on a sphere. Taking the initial direction to be along the z-axis, each small-angle collision causes a random step to be taken in the (v_x,v_y) plane.
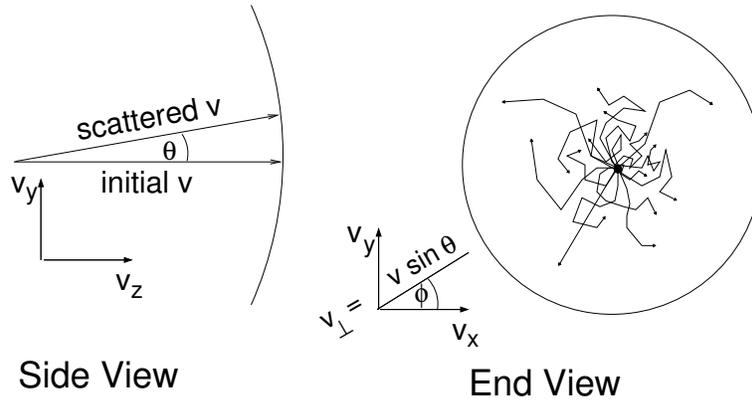
Figure 6.8: Multiple small-angle Coulomb collisions cause a diffusive "random walk" of the angle of the projectile velocity, or equivalently its perpendicular components.

Setting the large-angle collisions aside for a moment, we can treat the total angular scattering experienced by a projectile passing through a finite length of scattering path as the result of many small scatterings each of which has random direction and magnitude, governed by the fact that $\cot(\chi_c/2) = b/b_{90}$ (eq 6.17). Although we cannot calculate for any individual projectile what its final angle will be, we can treat the whole process statistically, by presuming there to be many small-angle scatterings. Actually, this calculation requires not the Rutherford differential cross-section per unit solid-angle Ω, but the differential cross-section per unit scattering angle χ_c,

$$\frac{d\sigma}{d\chi_c} = \left|\frac{db}{d\chi_c}\right|\frac{d\sigma}{db} = \frac{b_{90}}{2}\csc^2(\chi_c/2)\,2\pi b = \pi b_{90}^2\csc^2(\chi_c/2)\cot(\chi_c/2) \quad ,$$ (6.57)

which result is obtained immediately from our previous formulae.

The *mean* scattering angle experienced by the projectiles is always zero, because there is equal probability of scattering at negative and positive angles; the scattering is isotropic

in the (v_x, v_y) plane. The spread of scattering angles is quantified by the *mean square* scattering angle, which is non-zero and can be evaluated as follows. Succeeding collisions are statistically independent of each other. The final value of v_x is given by the sum of the steps in v_x at each of the individual collisions. (Similarly for v_y.) We therefore make use of the basic statistical theorem that the variance (which is the mean-square value for a zero-mean random variable) of the sum of independent random variables is the sum of the variances. We perform this sum by dividing the collisions into appropriate ranges of scattering angle $d\chi_c$ and azimuthal angle $d\phi$. The number of steps per unit path length belonging in these ranges is

$$dN = n_2 \frac{d\sigma}{d\chi_c} d\chi_c \frac{d\phi}{2\pi} \tag{6.58}$$

and the change in v_x that such collisions cause is

$$\delta v_x = (m_r/m_1) v_0 \sin \chi_c \cos \phi \quad . \tag{6.59}$$

Here, the quantity $(m_r/m_1) v_0$ is the initial (and final) speed of the incident particle (1) in the center-of-mass frame. Consequently, the total variance of v_x per unit path length arising from all possible types of collisions is

$$\frac{d\langle v_x^2 \rangle}{d\ell} = \int (\delta v_x)^2 dN = \int \int n_2 v_0^2 (m_r/m_1)^2 \sin^2 \chi_c \cos^2 \phi \frac{d\sigma}{d\chi_c} d\chi_c \frac{d\phi}{2\pi} \quad . \tag{6.60}$$

Performing the integration over azimuthal angle, ϕ, and substituting for the differential cross-section from eq (6.57) we get

$$\begin{aligned}
\frac{d\langle v_x^2 \rangle}{d\ell} &= \frac{1}{2} n_2 v_0^2 (m_r/m_1)^2 \int \sin^2 \chi_c \frac{d\sigma}{d\chi_c} d\chi_c \\
&= \frac{1}{2} n_2 v_0^2 (m_r/m_1)^2 \pi b_{90}^2 \int \sin^2 \chi_c \csc^2(\chi_c/2) \cot(\chi_c/2) d\chi_c . \tag{6.61}
\end{aligned}$$

The final integral may be transformed using trignometric identities, becoming

$$\int \sin^2 \chi_c \csc^2(\chi_c/2) \cot(\chi_c/2) d\chi_c = 8 \int \frac{1}{s} - s \, ds \qquad (s \equiv \sin \chi_c/2). \tag{6.62}$$

The upper limit of the integral is $s = 1$. The singularity of this expression at zero lower limit of s shows again the now-familiar need for a cut-off of the collision integral at large impact-parameter (small χ_c or s). That cut-off and eq(6.19) make the value of the integral $8(\ln |b_{\max}/b_{90}| - \frac{1}{2})$, where $b_{\max}$ is the maximum impact parameter, and the $\frac{1}{2}$ term should be dropped since it is an artifact of the approximation implied by our use of eq(6.59). In the case of scattering by a plasma, the relevant impact-parameter cut-off is the length beyond which the collective interactions in the plasma screen out the electric field of individual nuclei. This distance is called the Debye length. When the scattering is from neutral atoms, the relevant cut-off length corresponds to the size of the atom, because for impact parameters larger than the atom the projectile sees the whole atom, neutral because of its electrons, rather than a bare nucleus.

An identical treatment governs the y-component v_y, and consequently the square of the total transverse velocity $v_\perp^2 = v_x^2 + v_y^2$ evolves as

$$\frac{d\langle v_\perp^2 \rangle}{d\ell} = n_2 v_0^2 (m_r/m_1)^2 \, \pi b_{90}^2 \, 8 \ln |b_{\max}/b_{90}| \quad , \tag{6.63}$$

with $b_{\max}$ approximately the size of the atom. For small angles $\theta \approx v_\perp/v$ and so this equation can be written in terms of the angle of the scattered velocity direction:

$$\frac{d\langle \theta^2 \rangle}{d\ell} = n_2 \, (m_r/m_1)^2 \pi b_{90}^2 \, 8 \ln |b_{\max}/b_{90}| \quad . \tag{6.64}$$

After a finite path length ℓ, there is a distribution of $\mathbf{v}_\perp$ with variance

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \ell n_2 v_0^2 (m_r/m_1)^2 \, \pi b_{90}^2 \, 4 \ln |b_{\max}/b_{90}| \quad , \tag{6.65}$$

which we assume is still small compared to v_0^2 so that small-angle approximations remain valid. Because this distribution arises from many independent scatterings, it becomes Gaussian (following the Central Limit theorem of statistics):

$$f(v_x, v_y) = \left(\frac{1}{2\pi \langle v_x^2 \rangle} \right) \exp \left\{ -\frac{(v_x^2 + v_y^2)}{2\langle v_x^2 \rangle} \right\} \quad , \tag{6.66}$$

with $\langle v_x^2 \rangle$ given by eq(6.65). We may alternatively regard the Gaussian shape as arising because the particle distribution is experiencing a *diffusion* of velocity from an initial localized distribution (delta function) at $\mathbf{v}_\perp = 0$. The solution of the diffusion equation in this case is this Gaussian.

The maximum impact parameter (minimum χ_c) is determined by the shielding of the nucleus by its atomic electrons. Only for impact parameters small compared to the atom size, will the projectile see the bare nucleus because then it penetrates deep inside the electron shielding cloud. So b_{max} is approximately the radius of the electron cloud surrounding the nucleus. This is generally taken to have a characteristic size approximately[5] $a_0/Z^{1/3}$.

There is no mathematical compulsion to cut off the upper limit of the χ_c integral short of $\chi_c = \pi$, that is $s = 1$. However, if very energetic particles are involved, the value of b_{90}, which is inversely proportional to particle energy, becomes very small, eventually so small that it is smaller than the size of the nucleus. In that case, the large-angle scattering is affected by the structure of the nucleus itself and the upper limit is affected. Of course, this is the basis for experimental high-energy physics investigations of nuclear structure by electron scattering, but it requires electron energies greater than roughly $Ze^2/(4\pi\epsilon_0 r_n)$ ($\approx Z$ MeV), where r_n is the nuclear radius, of order 10^{-15} m, and Z its nuclear charge.

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[5] See M.Born "Atomic Physics" 8th ed., Blackie p199, for a derivation of the Thomas-Fermi distribution of electron density around an atom based on the Pauli exclusion principle and a continuum approximation.

6.4 Summary

Collisions of charged particles are governed by the long range Coulomb force. The range of that force is limited by one of several different processes, depending on the exact physical situation to a maximum impact parameter b_{max}. A minimum impact parameter for the process is also needed if approximations such as that the collision has a straight-line trajectory are made, or if quantum effects are important. Table 6.1 gives a summary of the situations discussed.

	Impact parameters		Stopping Power (per electron)
Collision Type	$b_{\min}$	$b_{\max}$	$\ln \Lambda = B/Z$
Classical Coulomb	b_{90}	$\gamma v_0/\omega$	$\ln \frac{\gamma v_0}{\omega b_{90}}$
Classical energy loss to atoms			$\ln\left\|\frac{1.123\gamma v_0}{\langle\omega\rangle b_{90}}\right\| - \frac{v_0^2}{2c^2}$
Quantum ion loss to atoms	$\sim \hbar/\gamma mv$	$\gamma v_0/\omega$	$\ln\left\|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right\| - \frac{v_0^2}{c^2}$
Corrected for inner shell effects			$\ln\left\|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right\| - \frac{v_0^2}{c^2} - C_k/Z$
Quantum electron loss to atoms	$\sim \hbar/\gamma mv$	$\gamma v_0/\omega$	$\ln\left\|\left(\frac{\gamma+1}{2}\right)^{1/2}\frac{(\gamma-1)m_e c^2}{\hbar\langle\omega\rangle}\right\| - \frac{v_0^2}{2c^2}$
Density effect (non-rel. plasma)	b_{90}	v_0/ω_p	$\ln\left\|\frac{v_0}{\omega_p b_{90}}\right\|$
Angular scattering from nucleus	b_{90}	$\sim a_0/Z^{1/3}$	$\ln\left\|a_0/Z^{1/3}b_{90}\right\|$

Table 6.1: Summary of collision calculations.

In collisions of the projectile particle 1, initial velocity v_0, with particles of type 2, density n_2, the rate of loss of kinetic energy K per unit pathlength ℓ is given by

$$\frac{dK}{d\ell} = -K \ n_2 \pi b_{90}^2 \frac{m_r^2}{m_1 m_2} 8\ln\Lambda = -n_2\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_2 v_0^2}\ln\Lambda$$

and the angular scattering from nuclei by

$$\frac{d\langle v_\perp^2\rangle}{d\ell} = v_0^2 \ n_2 \pi b_{90}^2 \frac{m_r^2}{m_1^2} 8\ln\Lambda = n_2\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{8\pi}{m_1^2 v_0^2}\ln\Lambda$$

with the $\ln\Lambda$ values indicated. See eqs(6.18) and (6.3) for other definitions.

Chapter 7

Radiation from Charged Particle Interaction with Matter

7.1 Bremsstrahlung

When charged particles collide, they accelerate in each other's electric field. As a result, they radiate electromagnetic waves. This type of radiation occurs when a fast electron slows down by collisions, and so it has acquired the German name Bremsstrahlung ("braking radiation").

7.1.1 Radiation in Collisions, Non-relativistic.

We have analysed collisions of charged particles in some detail in previous chapters, ignoring the possibility of radiation. The orbit of the projectile is, classically, a hyperbola. However, as an approximation, albeit one that will break down if the impact parameter, b, is small enough, we can ignore the curvature of the orbit and take the collision to occur with the projectile travelling along a straight line. This "straight-line-collision" approach we adopted previously as an approximation for calculating the energy transfer to a simple harmonic oscillator in a collision. Our present approach follows a parallel argument.

   As it passes the target, the projectile experiences the field of the target, which accelerates it. When the projectile is far away from the target, either before or after the collision, the acceleration becomes negligible. Therefore, the projectile has experienced an "impulse", a brief period of acceleration. We can estimate the duration of that impulse as being the time it takes the projectile to travel a distance of approximately b, namely $\tau = b/v_0$, where v_0 is the incoming projectile velocity. On average the impulse is perpendicular to the projectile velocity.

   The total energy radiated in this impulse is given by our previous formula (4.80) for the instantaneous radiated power by an accelerated charge,

$$P' = \frac{q_1^2}{4\pi\epsilon_0}\frac{2}{3c}\frac{\dot{v}^2}{c^2} \quad , \tag{7.1}$$

integrated over the duration of the impulse, τ. Taking the characteristic value of the acceleration as given by the electric field force at the closest approach,

$$\dot{\mathbf{v}} = E/m_1 = \frac{q_1 q_2}{4\pi\epsilon_0 b^2 m_1} \quad , \tag{7.2}$$

we derive an estimate of the radiated energy

$$W \approx P'\tau \approx \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 b^4} \frac{b}{v_0} = \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 v_0} \frac{1}{b^3} \tag{7.3}$$

This is the energy radiated in a single collision with impact parameter b. To obtain the energy radiated per unit length we multiply by the density of targets and integrate over impact parameters to obtain

$$\frac{dW}{d\ell} = n_2 \int \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 v_0} \frac{1}{b^3} 2\pi b \, db = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_1^2 v_0} \left[\frac{1}{b}\right]_{b_{min}}^{b_{max}} \quad . \tag{7.4}$$

Notice that in this case, there is no need to invoke an upper limit to the integration, b_{max}. We can perfectly well let b_{max} tend to infinity without any divergence of the integral. The same is not true of the lower limit. We will either have to invoke the limit on the classical impact parameter, b_{90}, where our straight-line approximation breaks down, or, more likely the usual quantum limit where the wave nature of the projectile becomes important, at

$$b_{min} = b_q = \frac{\hbar}{m_1 v_0} \quad . \tag{7.5}$$

With this quantum cut-off for b_{min} and infinity for b_{max}, the energy radiated becomes

$$\frac{dW}{d\ell} = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_1 \hbar} \quad . \tag{7.6}$$

7.1.2 Bremsstrahlung from light or heavy particles

So far we have treated the collision maintaining generality in the projectile and targets but have considered the radiation only from the projectile. Now we need to discuss what types of collisions give rise to significant bremsstrahlung. Equation (7.6) helps this discussion.

First we see that the projectile velocity does not enter into the formula. The projectile *mass*, however, is a very important effect. Light projectiles like electrons or positrons are far more efficient radiaters (by the inverse mass ratio) than protons or heavy nuclei, because their acceleration is so much greater.

That said, however, we realize that if a heavy projectile is colliding with a free electron target, then the electron target will experience an acceleration and give rise to radiation. This radiation from the target-particle acceleration is given by the same expression as before except with the charge and mass of the particles exchanged:

$$\frac{dW}{d\ell} = n_2 \frac{q_2^4 q_1^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_2 \hbar} \quad . \tag{7.7}$$

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Second, concerning targets, there are two effects that tend to cause the nuclei to dominate as targets in producing bremsstrahlung. The first effect is plain in eq (7.6). It is that the radiation is proportional to $q_2^2 \propto Z^2$, which for heavy atoms is a factor Z larger than the increase in the radiation caused by there being Z electrons per atom. The second effect that causes electron-electron collisions to be inefficient in producing bremsstrahlung is that the radiated electric fields caused by accelerations of the projectile electron and the target electron cancel each other.
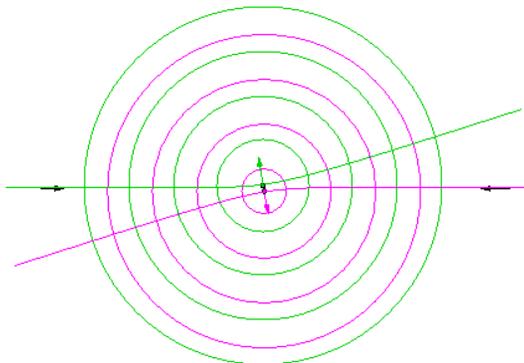

Figure 7.1: Electron-electron bremsstrahlung radiation wavefronts are out of phase and interfere destructively when the collision is close compared with the wavelength.

In electron-electron collisions, the accelerations of the projectile and the target are equal and opposite. Therefore they tend to give rise to equal and opposite radiated electric fields, which need to be coherently added together to obtain the total field. The far fields will actually cancel provided that there is only a small phase difference arising from the difference in the exact positions of the projectile and target electrons. That phase difference is roughly kb, where k is the relevant wave-number of the emitted radiation, and b is the impact parameter. However, the characteristic wave-number is given by

$$k = \frac{\omega}{c} \approx \frac{1}{c\tau} \approx \frac{v_0}{cb} \quad . \tag{7.8}$$

Therefore $kb \approx v_0/c$, in other words, the contributions from the projectile and target will cancel because $kb \ll 1$ if the incoming velocity is substantially less than the speed of light. Electron-electron bremsstrahlung is important only for relativistic electrons. Notice, though, that electron-positron bremsstrahlung does *not* produce this field cancellation, so it can be significant even in the non-relativistic case.

For the predominant case of electron-nucleus bremsstrahlung we can write eq (7.6) using the definitions of the fine structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c$ and the classical electron radius $r_e = e^2/4\pi\epsilon_0 m_e c^2$ as

$$\frac{dW}{d\ell} = n_2 Z^2 m_e c^2 \alpha \frac{4\pi}{3} r_e^2 \quad . \tag{7.9}$$

7.1.3 Comparison of Bremsstrahlung and Collisional Energy Loss

The question now arises of the relative importance of bremsstrahlung in calculating the energy loss of an energetic particle in matter. This is determined by the ratio of the radiated energy per unit length, eq(7.6), to the collisional energy loss, eq(6.45). For non-relativistic bremsstrahlung from collisions with nuclei, so that $n_2 = n_a$, this ratio is

$$\left| \frac{dW}{dK} \right| = Z_1^2 Z_a \alpha \frac{m_e v_0^2}{m_1 c^2} \frac{1}{3 \ln \Lambda} \quad , \tag{7.10}$$

using the definition of the fine structure constant, α, and denoting the atomic number of the nuclei as Z_a.

We see immediately, that bremsstrahlung in non-relativistic collisions is *never* an important contributor to the total energy loss, because even for electron collisions with the heaviest elements, $Z_2 \alpha \sim 92/137 \approx 0.67$ and dW/dK is much smaller than one because of the factors v_0^2/c^2 and $1/3 \ln \Lambda$.

If the projectile is a heavy particle, then the radiation from nuclear collisions is totally negligible, because of the mass ratio. One might be concerned then about radiation arising from the acceleration of the atomic electrons by the passing heavy particle. However, this can never exceed the energy transferred to the electrons in the collision, since the acceleration transfers the collisional energy as well as giving rise to radiation. Formally taking the ratio of eq(7.7) to the collisional loss we get the same expression as eq(7.10) except with m_1 replaced by the electron mass, thus confirming that bremsstrahlung is negligible in non-relativistic energy loss of heavy particles as well as electrons.

We shall see, nevertheless, that electron-nucleus bremsstrahlung can become important for relativistic electrons.

7.1.4 Spectral Distribution

We may want to calculate the spectrum of the electromagnetic radiation. It arises as a result of the impulse shape. For a single collision, the radiation's frequency spectrum will reflect the frequency spectrum of the impulse. An infinitely sharp impulse has a uniform frequency spectrum out to infinite frequency. This accelerating impulse has a duration $\tau \approx 2b/v_0$, and consequently has an approximately uniform spectrum only out to a cyclic frequency $\nu \approx 1/2\tau$. (The FWHM of Fourier transform of a square pulse τ is $\Delta\nu \approx 1/\tau$.)

Return therefore to the expression (7.3) for the radiated energy in a single collision with impact parameter b. This energy is spread over a total spectral width of approximately $1/2\tau$ so the energy spectral power density is

$$\frac{dW}{d\nu} \approx W 2\tau = \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{8}{3c^3} \frac{1}{m_1^2 v_0^2 b^2} \quad . \tag{7.11}$$

This is the energy spectrum radiated in a single collision of specified impact parameter. If we want to obtain the energy radiated per unit length, then as usual, we need to multiply by

the target density and integrate $2\pi b\,db$ over all impact parameters, which gives a logarithmic dependence:

$$\frac{d^2W}{d\ell d\nu} = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{16\pi}{3c^3} \frac{1}{m_1^2 v_0^2} \ln\left|\frac{b_{max}}{b_{min}}\right| \quad . \tag{7.12}$$

The b_{min} will arise because of the wave nature of the projectile, provided that the corresponding $b_{min} = \hbar/m_1 v_0$ is greater than b_{90}. For any fixed value of the photon frequency, ν, the maximum impact parameter at which this formula is appropriate is that parameter for which $\tau = b/v_0 \approx 1/2\nu$, since, as we have already discussed, for larger values of b the power spectrum falls off rapidly by virtue of the Fourier spectrum of the time variation of the electric field. Thus

$$\Lambda = \frac{b_{max}}{b_{min}} \approx \frac{m_1 v_0 v_0}{\hbar 2\nu} = \frac{\pi m_1 v_0^2}{\hbar\omega} \quad ,$$

Actually, since some energy and momentum is carried away by the photon radiated, the speed is not simply v_0 both before and after the collision. We could recognize that fact by substituting the average value of the velocity $\frac{1}{2}\{v_0 + \sqrt{[2(K - \hbar\omega)/m_1]}\}$ instead of v_0 in this logarithmic argument, where K is the initial kinetic energy. If we replace π with 2 in the logarithmic argument and also multiply the main coefficient by 2 (arbitrarily: the above was only an estimate), we obtain:

$$\frac{d^2W}{d\ell d\nu} = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{32\pi}{3c^3} \frac{1}{m_1^2 v_0^2} \ln\left|\frac{(\sqrt{K} + \sqrt{K - \hbar\omega})^2}{\hbar\omega}\right| \quad . \tag{7.13}$$

This expression is precisely what is obtained by a non-relativistic quantum mechanical calculation based on the Born approximation, first performed by Bethe and Heitler, 1934.

7.1.5 Bremsstrahlung from Relativistic Electrons

It is not straightforward to obtain estimates for bremsstrahlung from relativistic electrons. A key reason is that since the photon energy emitted extends from zero up to the electron's incident energy, we have to deal with photons having energies comparable to the electron rest mass or more, and hence carrying away momentum that is critical in the scattering process. One way to think about this process is to regard bremsstrahlung as the scattering of "Virtual Photons" associated with the field of the nucleus.

This approach, also known as the Weizsäcker-Williams method, after its earliest proponents, considers bremsstrahlung in the frame of reference in which the electron is stationary, and the ion moves past the electron. The electron feels a time-varying electric field of the ion, whose spectrum we have already discussed in the context of collisional energy transfer and the oscillator strength. This time-varying field (at least for velocities near the speed of light) can be approximated as a spectrum of plane waves. These are the virtual quanta.

The virtual quanta encounter the (initially) stationary electron. They can then be scattered by it, by the process of Compton scattering. In just the same way as photon momentum alters the Compton scattering process relative to the non-relativistic Thomson scattering, the bremsstrahlung is affected by the photon momentum and electron recoil. Because the
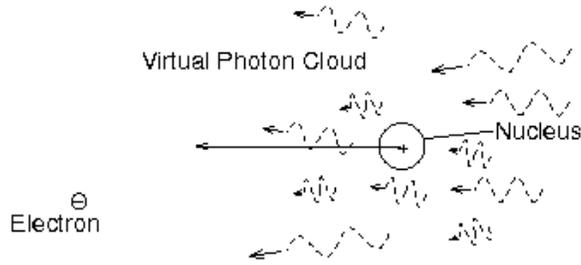
Figure 7.2: In the rest frame of the electron, the electric field of the nucleus is regarded as a "cloud" of virtual photons with a spectrum of energies, which scatter from the electron.

total Compton scattering cross-section falls off inversely proportional to the photon energy for energetic photons, [and in fact $\sigma_c \approx \sigma_T (3/4)(m_e c^2)/(\hbar\omega)$ for $m_e c^2 \ll \hbar\omega$, Jackson p 697], in this frame of reference, the scattered virtual photons (bremsstrahlung photons) are predominantly such that $\hbar\omega < m_e c^2$.

Recall that the virtual photon energy spectral density is essentially independent of the velocity [see section on straight-line collisions].

For relativistic velocities, the rate of scattering of photons of all energies is thus roughly constant, independent of collision energy, because it consists of a constant rate (of order the Thomson cross-section) of scattering of a constant photon spectral density up to a constant spectral limit ($m_e c^2$). The relativistic Doppler effect upshifts the majority of these photons in the laboratory frame to much higher energies, producing a spectrum extending up to $\gamma m_e c^2$, the electron energy in the lab frame. As illustrated schematically in figure 7.3. The *energy* loss rate is thus approximately proportional to the collision energy, because it consists of a constant rate of photon scattering but with energies on average proportional to the collision energy, $\gamma m_e c^2$.
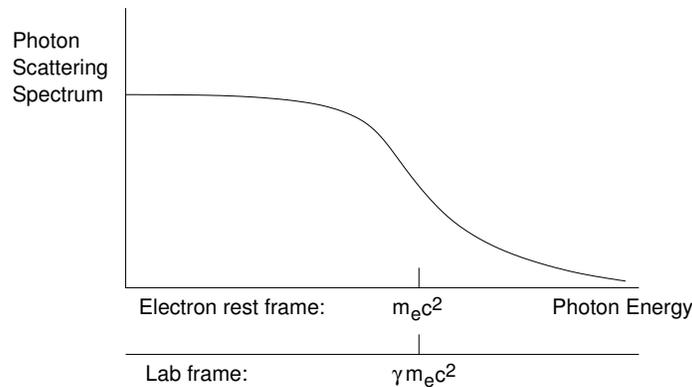

Figure 7.3: Photon scattering spectrum in the lab frame can be thought of approximately as a spectum flat to $m_e c^2$ in the electron rest frame, Doppler upshifted to $\gamma m_e c^2$ in the lab frame.

This qualitative argument indicates that we should expect the bremsstrahlung energy-loss

spectrum in the bulk of the relevant photon energy range to have a value given approximately by the same formula as the non-relativistic case (7.13), although with a different value of the logarithmic factor. The full relativistic formula can be written [Jackson eq 15.34]

$$\frac{d^2W}{d\ell d(\hbar\omega)} \approx n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{16}{3c^3\hbar} \frac{1}{m_1^2 v_0^2} \ln\left|\frac{2\gamma\gamma' m_e c^2}{\hbar\omega}\right| = n_2 Z_1^4 Z_2^2 \alpha r_e^2 \left(\frac{m_e}{m_1}\right) \frac{16}{3} \left(\frac{m_e c^2}{m_1 v_0^2}\right) \ln\Lambda \quad,$$
(7.14)

where $\gamma' = \gamma - \hbar\omega/m_e c^2$ is the relativistic gamma factor of the electron after the photon has been emitted, and $v_0 \approx c$, since this is a relativistic collision.

It is possible to write a universal expression for the photon energy spectrum per unit length, applicable for all energies. The quantum-mechanical Born-approximation calculations for electron projectiles yields this expression in the form [Heitler p 250]

$$\frac{d^2W}{d\ell d(\hbar\omega)} = n_2 Z_2^2 \alpha r_e^2 \frac{\gamma}{\gamma-1} B \quad,$$
(7.15)

where B is a dimensionless function of the ratio $\hbar\omega/m_e c^2(\gamma-1)$, that replaces the factor $(16/3)\ln\Lambda$. It is dependent on collision energy (i.e. $(\gamma-1)m_e c^2$) and photon energy, but weakly so. It has a value of order 15 over most of the photon spectrum. One can readily verify that this expression has the correct scaling with velocity at both low and high electron energy.

The magnitude of the cross-section is given by the term

$$\alpha r_e^2 = 0.580 \times 10^{-31} \text{ m}^{-2} = 0.580 \text{ millibarn} \quad.$$
(7.16)

(A barn is 10^{-28} m^{-2}).

7.1.6 Screening and Total radiative loss

We need to account for the screening of the nuclear potential by surrounding electrons of the atom when the collisions are distant.

The "Thomas-Fermi" potential is an approximation to the screened nuclear potential that can be approximated as

$$\phi = \frac{Ze}{4\pi\epsilon_0 r} \exp(-r/a) \quad,$$
(7.17)

with the characteristic length $a \approx 1.4a_0 Z^{-1/3}$. This form of screening is identical to what applies to Coulomb collisional energy loss etc.

It is most important at low photon energy (relative to the incident energy) because the distant collisions are most effective there. It reduces the cross-section (or power radiated) because it essentially lowers the maximum effective impact parameter to $\sim a$.

Estimates of the screening effect can be obtained by putting b_{max} equal to a instead of v_0/ω, resulting in a logarithmic factor that for non-relativistic collisions is

$$\Lambda = \frac{b_{max}}{b_{min}} \approx \frac{m_1 v_0 a}{\hbar} = \frac{m_1 v_0 1.4a_0 Z^{-1/3}}{\hbar} = \left(\frac{1.4\beta}{\alpha Z^{1/3}} \frac{m_1}{m_e}\right) \quad,$$
(7.18)

where the final form follows from

$$\frac{\hbar}{a_0} = m_e c \alpha \tag{7.19}$$

Actually screening effects are most important not for non-relativistic collisions but for relativistic collisions. For relativistic collisions, we replace the characterisic maximum impact parameter $2\gamma\gamma'c/\omega$ with a if a is smaller, so that screening is important. It will be if

$$\left(\frac{\omega}{2\gamma^2 c}\right)\left(\frac{1.4a_0}{Z^{1/3}}\right) < 1 \tag{7.20}$$

This inequality will apply over the entire frequency range up to the maximum possible photon energy $\hbar\omega = \gamma m_1 c^2$ if the incident energy satisfies:

$$\frac{m_1 c^2}{2\gamma c\hbar}\frac{1.4a_0}{Z^{1/3}} = \frac{0.7}{\alpha\gamma Z^{1/3}}\frac{m_1}{m_e} < 1 \quad , \tag{7.21}$$

using eq (7.19) again. This criterion is $\gamma > 196/Z^{1/3}$ for electron projectiles. When it is satisfied, the collisions are said to be in the range of "complete screening", and the logarithmic factor becomes $\ln\Lambda \approx \ln(233/Z^{1/3})$. [Jackson p722, although our calculation would make it $\ln(192/Z^{1/3})$].

For non-relativistic electrons, the radiative energy loss is always negligible compared with the collisional loss. This is not the case for strongly relativistic electrons because the total bremsstrahlung power loss, for the roughly constant spectral power, is proportional to the total width of the spectrum, i.e. to the collision energy.

Taking the completely screened cross-section case, in which the logarithmic term and $\gamma/(\gamma - 1)$ are approximately constant, the total spectrally integrated energy loss rate is given by

$$\frac{dW}{d\ell} = \int \frac{d^2 W}{d\ell d(\hbar\omega)} d(\hbar\omega) = n_2 Z_2^2 \alpha r_e^2 \left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right| \gamma m_e c^2 \tag{7.22}$$

So writing $K = \gamma m_e c^2$ for the total electron energy, we get a slowing down equation

$$-\frac{dK}{d\ell} = K n_2 Z_2^2 \alpha r_e^2 \left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right| \tag{7.23}$$

If we compare this to the slowing down rate due to collisional effects (excluding bremsstrahlung) we find that these rates, whose dependence on the nuclear charge, Z are different, are equal when $\gamma \approx 200$ for air and $\gamma \approx 20$ for lead.

When bremsstrahlung loss predominates over collisional loss, the energy is sufficient for the screening to be complete. Then the slowing down rate is constant. That is, the energy loss equation reduces approximately to

$$-\frac{dK}{d\ell} = K/\lambda \tag{7.24}$$

with exponentially decaying solutions $K \propto \exp(-\ell/\lambda)$ having characteristic length:

$$\lambda = \left[n_2 Z_2^2 \alpha r_e^2 \left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right|\right]^{-1} \tag{7.25}$$

The expressions most quoted are slightly different [Heitler, and subsequently Evans] replacing as follows in the completely screened limit:

$$\left(\frac{16}{3}\right)\ln\left|\frac{233}{Z^{1/3}}\right| \to 4\ln\left|\frac{183}{Z^{1/3}}\right| + \frac{2}{9} = B \tag{7.26}$$

although the difference is small, within uncertainties of the whole approximate approach.

7.1.7 Thick target Bremsstrahlung.

Remarks not typed up.

7.2 Čerenkov Radiation

Maxwell's equations with a dielectric medium:

$$\nabla \wedge \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{7.27}$$

The current consists of partly the medium polarization

$$\mathbf{j}_{\text{medium}} = \frac{\partial \mathbf{P}}{\partial t} \tag{7.28}$$

and partly "external" currents, $\mathbf{j}_x$, like the particle moving through it. We combine the polarization current into the $\frac{\partial \mathbf{E}}{\partial t}$ term, using the standard relationship $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, to get

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}_x + \mu_0 \frac{\partial \mathbf{D}}{\partial t} = \mu_0 \mathbf{j}_x + \frac{1}{c^2}\frac{\partial}{\partial t}(\epsilon \mathbf{E}) \tag{7.29}$$

where ϵ = dielectric constant = relative permittivity. Eliminate $\mathbf{B}$:

$$-\nabla \wedge (\nabla \wedge \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}_x}{\partial t} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\epsilon \mathbf{E}) \tag{7.30}$$

or

$$-\nabla(\nabla.\mathbf{E}) + \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\epsilon \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}_x}{\partial t} \tag{7.31}$$

This is now a wave-equation but with a source on the right-hand side. A helpful way to think about Cerenkov radiation is then to regard the current of the swift particle as coupling to oscillators consisting of plane waves propagating with wave-vector $\mathbf{k}$ and frequency ω. Because of the dielectric medium the wave "oscillators" satisfy $k^2 c^2 = \epsilon \omega^2$. This is the standard result that the refractive index of a transverse wave in a dielectric is

$$\frac{kc}{\omega} = \epsilon^{1/2} = N \tag{7.32}$$

The wave velocity is $\frac{\omega}{k} = c/\epsilon^{1/2}$, i.e. the waves travel slower than c. This allows the particle to couple to the oscillators <u>resonantly</u>. We saw previously (oscillator strength calculation)

that it is resonance that is required [$|E(\omega)|^2$ is what gives the energy transfer]. For resonance with a wave $\propto \exp i(\mathbf{k.x} - \omega t)$, we require a uniformly moving particle (i.e. one without an intrinsic oscillating frequency) to move such that the phase of the wave is constant at the particle. Particle position is $\mathbf{r} = \mathbf{v}t$ (+ constant) so resonance is

$$\text{constant} = \mathbf{k.r} - \omega t = (\mathbf{k.v} - \omega)\, t \tag{7.33}$$

i.e. $\mathbf{k.v} = \omega$. So if we choose a specific frequency ω, we need to satisfy simultaneously

1. $k = \frac{\omega}{c}\epsilon^{1/2}$ (wave dispersion relation)

2. $\mathbf{k.v} = \omega$ (resonance with particle)

Graphically: For a particle moving faster than wave phase velocity $(v > c/\epsilon^{1/2})$ solutions ex-
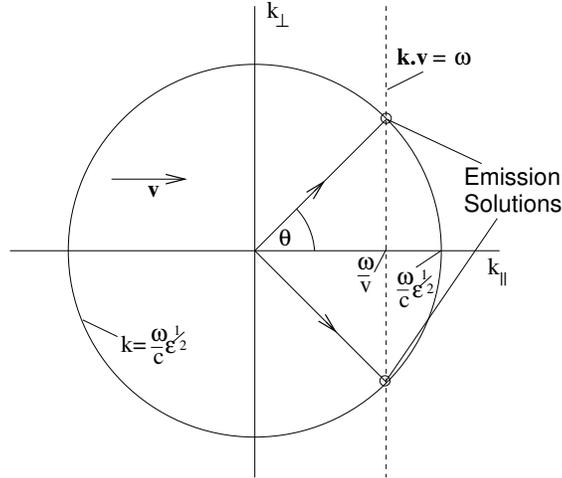

Figure 7.4: k-coordinates for satisfying resonance and the dispersion relation

ist because $\frac{\omega}{v} < \frac{\omega}{c}\epsilon^{1/2}$, otherwise not. Cerenkov radiation requires "superluminary" velocity. Also angle between $\mathbf{k}$ and $\mathbf{v}$ is simply given by

$$\cos \theta = \frac{\omega}{kv} = \frac{c}{v\epsilon^{1/2}} \tag{7.34}$$

If ϵ is independent of ω, the result is to form an optical "shock front" All electromagnetic wave fronts add coherently along the shock front, leading to a singularity. Actually if ϵ is $> \frac{c^2}{v^2}$ for all frequencies, then an infinite amount of energy is radiated per unit length. This is a reflection of the singularity at the shock front. The variation of ϵ with frequency is crucial for proper treatment of Cerenkov emission. Optical materials have refractive index that does vary with frequency (prism splits spectrum of white light). Resonance in atoms of medium is usually in UV. Radiation can occur for all frequencies up to the resonance (different resonance from Cerenkov) and down to the place where $\frac{v}{c} = \frac{1}{\epsilon^{1/2}}$. Variation $\epsilon(\omega)$ removes the singularity, gives a spectral variation and finite spectral range.
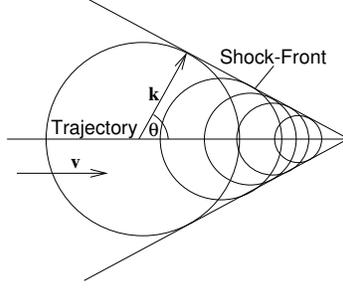
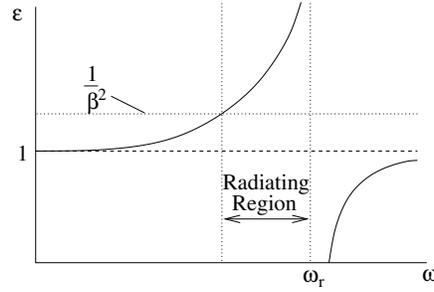Figure 7.5: Shock-Front arising from coherent addition of waves from all along the particle trajectory.

Figure 7.6: Typical variation of the relative permittivity of a transparent material.

7.2.1 Coupling Strength

We are interested in <u>transverse</u> waves

$$\mathbf{k}.\mathbf{E} \quad = \quad 0 \quad (\Rightarrow \nabla.\mathbf{E} = 0) \tag{7.35}$$

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\epsilon\mathbf{E}) \quad = \quad -k^2\mathbf{E} + \frac{\omega^2\epsilon}{c^2}\mathbf{E} = \mu_0\frac{\partial \mathbf{j}_x}{\partial t} \tag{7.36}$$

$\mathbf{E}$ is perpendicular to $\mathbf{k}$. If $\mathbf{v}$ is in x-direction, then, $\mathbf{k} = k(\cos\theta, \sin\theta)$

$$\mathbf{E} \quad = \quad E(-\sin\theta, \cos\theta, 0) \quad \text{or} \tag{7.37}$$

$$= \quad E(0, \quad 0, \quad 1) \quad \text{are possible polarizations} \tag{7.38}$$

But coupling to the wave is determined by the vector $\frac{\partial \mathbf{j}_x}{\partial t}$. For moving point particle $\mathbf{j}_x = q\mathbf{v}\delta(\mathbf{x} - \mathbf{v}t)$ which has only x-component. Hence

1. It does not couple at all to E_z polarization.

2. Coupling to the in-plane polarization, $\mathbf{E} = \mathbf{E}(\sin\theta, \cos\theta)$ is proportional to $\frac{\mathbf{E}.\mathbf{j}}{Ej}$ i.e. $\sin\theta$

Final point note that driver is $\frac{\partial \mathbf{j}_x}{\partial t}$ so that since the spectrum of j_x is flat, because it is a delta function in time, the spectrum of the drive term is $\propto (i)\omega$. All of this can be made
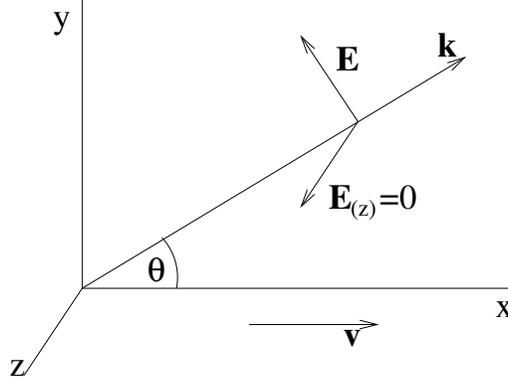
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Figure 7.7: Polarization of Čerenkov emission is purely in the plane of emission. Coupling to $\mathbf{E}_{(z)}$ is zero.

rigorous. The result is that the energy radiated per unit length of path is

$$\frac{dW}{d\ell} = \frac{q_1^2}{4\pi\epsilon_0}\frac{1}{c^2}\int_{\epsilon(\omega)>\frac{1}{\beta^2}}\left(1 - \frac{c^2}{v^2\epsilon(\omega)}\right)\omega\,d\omega \quad \text{[Frank, Tamm 1937]} \tag{7.39}$$

and we can identify the terms as

$$1 - \frac{c^2}{v^2\epsilon} = 1 - \cos^2\theta = \sin^2\theta \tag{7.40}$$

i.e. the <u>coupling dependence</u> on radiation <u>angle</u>. Squared because energy goes like the square of the electric field.

$$\omega \text{ arising from } \quad \frac{\partial}{\partial t}j_x. \tag{7.41}$$

This equation also gives the frequency spectrum of the radiated power (the integrand) but it is non-zero only for frequencies such that $\epsilon > \frac{c^2}{v^2}$ or $v >$ phase velocity $\frac{c}{\epsilon^{1/2}}$. Energy emitted per unit length is estimated by putting $1 - \frac{c^2}{v^2\epsilon}$ equal to an average value and so

$$\int_{\omega_1}^{\omega_2}\left(1 - \frac{c^2}{v^2\epsilon}\right)\omega\,d\omega \simeq \frac{1}{2}\left[\omega_2^2 - \omega_1^2\right]\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \tag{7.42}$$

where $\omega_{2,1}$ are the upper and lower limits of spectral region of emission.

$$\begin{aligned}
\frac{dW}{d\ell} &= \frac{q_1^2}{4\pi\epsilon_0}\frac{1}{c^2}\frac{1}{2}\left[\omega_2^2 - \omega_1^2\right]\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \\
&= \alpha\frac{\omega_2}{c}\frac{1}{2}\left[\hbar\omega_2 - \frac{\hbar\omega_1^2}{\omega_2}\right]\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \\
&\simeq \alpha\frac{\omega_2}{c}\frac{1}{2}\hbar\omega_2\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \quad \text{if } \omega_1 << \omega_2. \\
&= \alpha\frac{\pi}{\lambda_2}\hbar\omega_2\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \quad \left(\frac{\lambda}{2\pi} = \frac{c}{\omega}\right) \tag{7.43}
\end{aligned}$$

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The rough value of this energy per unit length can be estimated noting that the resonance (where $\bar{\epsilon} \to \infty$) in the optical response of glasses is generally near $\lambda_2 = 100$ nm $\Rightarrow \hbar\omega_2 = 12eV$, and near that resonance $\frac{c^2}{v^2\bar{\epsilon}} \to 0$ so

$$\frac{dW}{d\ell} \sim \alpha\pi\frac{1}{10^{-7}}.12 \text{ eV/m} = 2.7 \times 10^6 \text{eV/m} \tag{7.44}$$

This is tiny in comparison with the rate of loss of energy by other processes. The number of photons emitted per unit length is even easier

$$\frac{dN}{d\ell} \simeq \frac{q_1^2}{4\pi\epsilon_0}\frac{1}{\hbar c^2}\left[\omega_2 - \omega_1\right]\left[1 - \frac{c^2}{v^2\bar{\epsilon}}\right] \tag{7.45}$$

$$= \alpha 2\pi\left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right]\left[1 - \frac{c^2}{v^2\bar{\epsilon}}\right] \tag{7.46}$$

$$\simeq \alpha 2\pi\frac{1}{\lambda_2} \quad (\text{for } \omega_2 >> \omega_1). \tag{7.47}$$

[And the photon spectral distribution is

$$\frac{dN}{d\ell d\omega} = \alpha\frac{1}{c}\sin^2\theta = \alpha\frac{1}{c}\left[1 - \frac{c^2}{v^2\epsilon}\right]. \quad] \tag{7.48}$$

Rough estimate of photons (total) / length for $\lambda_2 \simeq 100$ nm:

$$\frac{dN}{d\ell} \simeq 5 \times 10^5 \text{ photons/m} \tag{7.49}$$

Optical range ($\lambda \simeq 400 - 600$ nm) contains about $(\frac{1}{4} - \frac{1}{6}) = 0.083$ times as many ($\times \sin^2\theta$) so can be as little as $\frac{1}{25}$ of this total ~ 20 photons/mm.

7.2.2 Energy Spectrum

$$\text{Energy Spectrum proportional to} \quad \omega\left(1 - \frac{c^2}{v^2\epsilon}\right) \tag{7.50}$$

is

1. broad and smooth.

2. larger at larger ω (smaller λ) "blue" because

   (a) ω factor
   (b) ϵ increase with $\omega \Rightarrow 1 - \frac{c^2}{v^2\epsilon}$ increases

Result: Bluish-White light. Observed by Marie Curie 1910. Studied in detail by Cerenkov 1935. Explained Frank & Tamm 1937 (classical). Used for detectors starting mid 1940s.